**Risk, Preference and Utility**

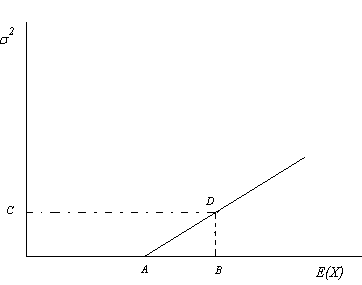
Financial decisions, very often, if not always, involve risky choices. Therefore, valuing risky opportunities is a major challenge for financial managers. If several alternative actions were comparable in the amount of risk they involved, the choice among them would have been easy. Most rational decision makers would agree on the correct choice: the one with the largest expected return (EMV). However, when the alternatives involve dissimilar risks, as they often do, the “correct” choice is not as straight forward. The maximum EMV criterion does not always describe the behavior of rational individuals because EMV is completely blind to risk and people pay attention to risk in making financial choices. To illustrate how EMV leads to uncharacteristic decision behavior, consider a risky situation where you are asked to invest $10,000 in a venture where there is one percent probability for a profit of $1,000,000 and 99% chance of a complete bust. The expected value of this investment is $100. But no sane individual would invest in this venture where losing $10,000 is a near certainty. Another example is a proposal to flip a fair coin for winning $10,001or losing $10,000. The expected value of this stake is $.50. But again most rational people would not risk their hard-earned $10,000 for an even chance of a $1 gain. Without loss of generality (since we are talking prudent business decision making) we will assume that risk is undesirable and that unless they are somehow compensated for it most decision makers would try to avoid risk as much as possible. In both examples above despite positive EMV, the disutility of the inherent risk sways most rational people away from the choice based on the EMV criterion.

To further illustrate the conceptual difficulty in the evaluation of risky decision alternatives stemming from the subjective nature of risk avoidance, consider an example where you put $1 to win $2 or nothing with even probabilities (EMV = $1). If risk is indeed undesirable we should expect rational people to be unwilling to pay the $1 for this lottery for they prefer their $1 to stay in their pocket rather than risking it on the lottery which has a risky expected pay-off of $1 (nothing or $2). However, many people will, as some of you might, take this gamble. Should we consider such people irrational? Not at all: those willing to take the gamble are being adequately compensated by the enjoyment and the thrill of the gamble. Thus the risk behavior of sane individuals is a bit more complex than simple risk avoidance—if the stakes are not too high they may even be risk seeking. The whole multi-billion dollar gambling industry is founded on marketing this enjoyment. As the stakes increases, however, the disutility of the risk begins to exceed the utility of the enjoyment and thrill of the gamble. We would find very few buyers for a $50,000 ticket for stakes of $100,000 or nothing with equal probabilities. The disutility of coming up empty and losing the ticket price of $50,000 would be too much for the thrill of the gamble to compensate for most people.

A decision maker in making his choice has to account and make provisions for the risks underlying the alternatives. There are two implicit components in his task: to quantify risk and to establish a mechanism by which the measured risk can be traded against expected return. Conceptually, the later is the more difficult step since the sufficient compensation for a given amount of risk depends on the risk-taking behavior of the decision maker—inherently a subjective matter. There are at least two alternative approaches one can take in measuring and trading risk: the well known mean-variance approach and utility theory. The rest of this note will briefly review the mean-variance concept but more fully develop the utility approach.

Mean-Variance Approach for Risk Analysis

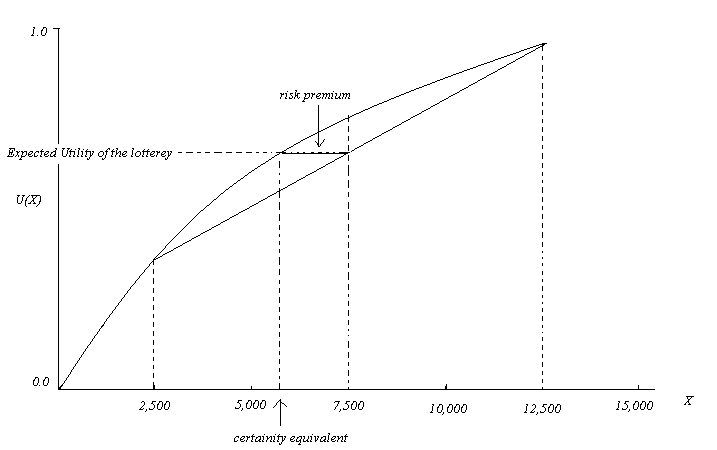
Any risk can be modeled as uncertain returns with estimated probabilities—or probability distributions. The risk of losing one’s home to a hurricane can be modeled by a two valued distribution of no loss versus the loss of the value of the house with estimated probabilities; the risk that an investor faces can be modeled as the probability distribution possible of returns (positive or negative). Thus, at least in theory, risk can be modeled as a property of the underlying probability distribution. The mean-variance approach to risk management uses the variance of the returns distribution as the measure of the magnitude of the risk: the bigger the variance, the more uncertain the returns are, the higher the risk is. Mean-variance approach assumes that every risk-averse decision maker has an “indifference curve” where increasing risk is compensated by higher mean returns as in the following graph. Indifference curve implies that the decision maker is valuing a risky opportunity by combining a measure of risk with a measure of return. The common measure of return is the EMV (mean return) and the common measure of risk is *2* thus the ‘value’ of the opportunity is EMV - *2* where  can be interpreted as the ‘unit cost of risk.’ graph depicts a risk-averse decision maker who considers point *A* (risk free)



return and *D* (representing a return of *B > A* and a risk of *C* > 0) to be of equal attractiveness. Although the indifference curve is drawn as a straight line here, a priori, there is no theoretical reason why it could not be concave or convex, implying different risk aversion behaviors. A couple of intuitively appealing concepts are readily apparent in this graph. For this decision maker, the difference between the returns of *B* and *A* may be defined as the *risk premium* for a risk of *C,* namely additional return needed to barely compensate the decision maker for assuming the risk of magnitude *C.* Furthermore, for the risky return at point *D* (return of *B* with risk *C*), or for any combination of returns and risks on the line, the sure return at *A* may be referred as the *certainty equivalent*.

Utility Approach to Risk Analysis

Assuming that the pay-offs of a decision situation (such as an investment) are expressed in monetary terms, this approach associates a cardinal (quantifiable) utility function for various amounts of money. The scale used to measure utility is arbitrary and is set by convention: commonly the worst outcome is associated with utility of 0 while best outcome with 1, any other monetary outcome between these two extremes are associated with a utility amount between 0 and 1. It is assumed that for rational decision makers the utility for money is a non decreasing function of amount of money *i.e*., no rational decision maker would prefer less money to more. A concave (convex) function represents a risk-averse (seeking) behavior, while a linear one models indifference to risk. To see this consider the example: you are offered for a price, a lottery ticket which will pay $2,500 or $12,500 with equal probabilities. The expected return (*EMV*) for the ticket is $7,500 (0.5\*2,500 + 0.5\*12,500). Would you pay $7,500 to own this ticket? If you are risk averse, you would not—for a risk-averse person the additional utility of making a profit of $5,000 ($12,500 – $7,500) does not quite balance the disutility of losing$5,000 ($7,500 - $2,500). However, there is a price less than $7,500 at which you might consider buying the ticket and assuming the risk. How much, at most, would you pay for this lottery ticket? For the sake of discussion, let’s say you would pay $6,000 (some one else might pay more or less); and no more. This means that you are indifferent between $6,000 for sure and the risky lottery *i.e*., they have the same utility for you. As before, let us define the $6,000 as the *certainty equivalent* (*CE*) of the lottery and the difference between the *EMV* and the certainty equivalent or $1,500 as the *risk premium (RP)*. As can be seen in the following graph a positive risk premium implies risk-aversion and thus a concave utility function. Such a utility function has the property that the marginal utility of additional wealth decreases as wealth increases. For someone with $20,000 annual income a $1,000 bump in salary would be vigorously celebrated while the same increase for someone making $200,000 would be barely noticed. Interestingly, diminishing marginal utility for money is the same phenomenon as risk avoidance.



The concept of certainty equivalent allows us to conceptualize any risky outcome (a probability distribution) as a series of simple binary lotteries as in the above example. Suppose the outcome of the decision had three outcomes rather than two, we could take any two of the outcomes and replace them with their certainty equivalent and thus end up with a binary lottery. This approach can be applied as many times as necessary to break any number of risky outcomes into a series of binary lotteries which are easier to analyze.

Before we go any further let us reiterate the purpose of trying to measure and trade risk and return. The object in both approaches—mean-variance and utility is to incorporate the risk as a factor in choosing among alternatives. This could be done either by substituting certainty equivalent (risk adjusted) return for the risk-blind *EMV* as the decision criterion; or equivalently using the criterion of maximum expected utility. To illustrate the later consider a decision alternative with returns of *X*1, *X*2, *X*3, . . ,*XN* with probabilities of *p*1, *p*2, *p*3, . . ,*pN*. While the expected monetary value of this alternative is: , the expected utility is .

Determining the utility function

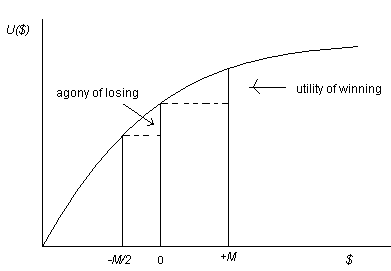
There are two main approaches for constructing the utility function of a decision maker. The first, as you saw in the text, is to elicit it from the decision-maker, by offering him lotteries consisting of the best and the worst outcomes. To briefly illustrate this process, in a particular decision situation where the best outcome is $12,500 and worst $0, suppose we arbitrarily set[[1]](#footnote-1) *U*($0) = 0 and *U*($12,500) = 1, namely we will assign utility values between 0 and 1 to outcomes in the range from $0 to $12,500. Say we want to find, for some decision maker, *U*($5000). First we find the break-even probability, *po* of winning in the lottery such that expected monetary value of the lottery of 0$ and $12,500 is exactly $5,000. Namely, we solve $5,000 = $0 \* (1- *po*) + $12,500 \* *po* for *po*. This yields *po* = 0.4. If the decision maker is risk indifferent, he should regard the lottery ($0, $12,500) with a winning probability of 0.4 as desirable as a sure $5,000 since by construction the lottery’s *EMV* is $5,000. If, however, he is risk averse, he should prefer the sure $5,000 to a risky $5,000 (*EMV* of the lottery). The lottery, however, becomes more attractive if the winning probability increases gradually beyond 0.4 closing the utility gap between the sure $5,000 and the lottery. Thus for some probability of winning better than 0.4, the decision maker should be indifferent between the lottery and the sure $5,000. Say, this happens for a specific decision maker for a probability of winning of 0.55 *i.e.*, the decision maker is as happy with the lottery if the probability of winning is 0.55 as he is with $5,000 for sure. This means that the expected utility of the lottery is equal to the utility of $5,000, or *U*($5,000) = (1-0.55)\* *U*($0) + 0.55\* *U*($12,500). By convention, *U*($0) = 0 and *U*($12,500) = 1. Substituting these in the equation we get *U*($5,000) = 0.55. This process can be repeated a number of times to obtain the utility of the decision maker for various amounts between $0 and $12,500 to approximate his entire utility function. Notice that the way the indifference lottery is set up implies that the certainty equivalent is $5,000 and since the *EMV* = (1 -0.55)\*0 + 0.55\* $12,500 = $6,875, the risk premium is $1,875.

The second approach is to use predetermined mathematical functions having properties appropriate to the decision maker in order to approximate his utility function. For instance *U*(*X*) =  has most of the properties of utility function of rational individuals*. U*(0) = 0 , *U*(*∞*) = *∞*; it is non decreasing and concave exhibiting the property of diminishing marginal utility of money, but undefined for negative monetary results. Two more commonly used mathematical functions for this purpose are the *negative exponential* utility function and the *logarithmic* utility function. These represent different risk behaviors and thus may be appropriate in different situations for different decision makers.

*The negative exponential utility function* has the property of modeling *constant* risk-aversion behavior. If a person wants the same risk premium for a binary lottery regardless of his current wealth he is said to be constantly risk-averse. In the example below the lottery is -$2,500 and $5,000 with some probability of winning. The four scenarios correspond to ending position (after the lottery) if the decision maker starts with various levels of ‘wealth.’ This decision maker’s risk premium for this lottery is independent of his starting ‘wealth,’ thus he is constantly risk averse. The negative exponential utility function,  has constant

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| --- | --- | --- | --- | --- |
| **Starting position** | **Ending position** | **EMV** | **CE** | **RP** |
| $0 | -$2,500 or $5,000 | $1,250 | $750 | $500 |
| 2,500 | 0 or 7,500 | 3,750 | 3,250 | $500 |
| 5,000 | 2500 or 10,000 | 6,250 | 5,750 | $500 |
| 7,500 | 5,000 or 12,500 | 8,750 | 8,250 | $500 |

risk-aversion property. Its only parameter, *R* is called the *risk tolerance constant* and determines how risk-averse the individual is (smaller values of *R* representing stronger risk-aversion). In this model, unlike the first method, constructing the utility function of a decision maker simply boils down to determining the decision maker’s personal *R,* the risk tolerance constant. The value of *R* is elicited from the decision maker by offering him binary lotteries whose outcomes are $*M* and –$*M*/2 with equal probabilities *e., g*. ($5 and -$2,5); ($10 and -$5); ($1,000 and -$500), etc. Notice that as *M* increases, the potential loss also increases. Due to diminishing marginal utility, for large values of *M*, the utility of gaining *M* will not be enough to compensate the decision maker for the loss of *M/2.*  Appropriate *R* for the decision maker is that *M* for which he is indifferent between taking or leaving the lottery. This is illustrated in the following graph where the utility of winning $*M* exactly equals the disutility of losing $*M*/2. For lotteries with values larger than *M,* the utility of winning will be less than the disutility of losing, and *vice versa*.



For this utility function, the certainty equivalents of risky returns can be calculated directly by applying the following formula:.

**Example**: If a decision maker’s *R* = $20,000, find the certainty equivalent and the risk premium for returns of -$2,500, $0 and $10,000 with probabilities of 0.4, 0.2 and 0.4 respectively. Note than this is a fairly risk tolerant decision maker willing to risk losing $10,000 for fifty percent chance of making $20,000.

*EMV* = 0.4\*-$2,500 + 0.2 \* $0 + 0.4\*$10,000 = $3,000.

*CE* = -20,000[ln (0.4*e* 2500/20,000 + 0.2*e* 0 + 0.4e -10000/20,000)]

= -20,000[ln (0.4\* 1.1331 + 0.2\* 1 + 0.4\* 0.6065)] = -20,000[ln(.8959)] = $2,199.69

*RP = EMV – CE* = $3,000 - $2199.69 = $800.31.

*The logarithmic utility function* models decreasing risk-aversion behavior. Most people tend to become less risk-averse as they become wealthier. Insurance buying behavior of many people illustrates this behavior. Younger and thus poorer people tend to opt for small deductibles in their auto insurance because it may be a real hardship to cover the deductible in case of an accident. However as the same people become more wealthy their ability to cover the same deductible increases making them more willing to take the risk of having to cover the deductible. In fact, when they become so wealthy that the potential loss from an accident is miniscule compared to their wealth, they may not even buy insurance. We see that many very large companies (with large net worth) opt for self insurance whereas smaller companies transfer many risks they may face to insurance companies. To illustrate this behavior let us look at the previous example lottery with various initial ‘wealth’ positions. As the initial position (wealth) increases the risk premium for the lottery decreases. Although there are many mathematical

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Starting position** | **Ending position** | **EMV** | **CE** | **RP** |
| $0 | -$2,500 or $5,000 | $1,250 | $750 | $500 |
| 2,500 | 0 or 7,500 | 3,750 | 3,300 | $450 |
| 5,000 | 2500 or 10,000 | 6,250 | 5,850 | $400 |
| 7,500 | 5,000 or 12,500 | 8,750 | 8,400 | $350 |

functions exhibiting this property, the logarithmic function  is frequently used in modeling the diminishing risk-averse behavior in financial markets. The function is defined for *X > -A* where *A* is interpreted as the decision maker’s entire net worth. It is undefined for losses exceeding *A* since such losses are tantamount to complete ruin. The risk aversion decreases *i.e*., decision maker’s risk premium decreases, as the he becomes richer. For this function the certainty equivalent of *n* discrete uncertain returns, *Xi* with probability *pi* uncertain returns is given by:



Example: if decision maker’s *A* = 20,000, find the certainty equivalent and risk premium of the returns in the above example.

*CE* = [(-2,500 + 20,000)0.4 (0 + 20,000)0.2 (10,000 +20,000)0.4] – 20,000

*CE* = [49.7984 \* 7.2478 \* 61.7801] – 20,000 = $2,298.196

*EMV* = $3,000 as before

*RP* = $3,000 – $2,298.17 = $701.83.

As we pointed out earlier, in the mean-variance approach a risky investment is valued by adjusting the mean returns for the risk or as *EMV* - *2*. Notice that a constant ** implies constant risk-aversion behavior and *2* is the risk premium that does not depend on *EMV*. It is interesting to note that the exponential utility function approximates  by 0.5*R.*  Also the approximation becomes better as the returns distribution approaches to a normal distribution. Thus the certainty equivalent of a normally distributed returns can easily be calculated as *CE = EMV* - *R2*.

A complete [example](riskexample.xls)

Consider the following three investment options: real estate, indexed fund and a new venture in nanotechnology. The returns to the investments are estimated depending on the state of the economy and are given as in the following pay-off table. The investor has $1M to invest and does not wish to invest in more than one option. The probabilities of Bearish, Neutral and bullish conditions in the economy are estimated to be 0.25, 0.35, 0.40.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Bearish | Neutral | Bullish |
| Real Estate (RE) | -2% | 3% | 9% |
| Index Fund (IN) | -5% | 0% | 15% |
| New venture(NV) | -25% | -2% | 30% |

1. *Calculate the EMV of all three and make a recommendation if the investor is risk-neutral.*

Converting the returns to monetary pay-offs (in $000)

*EMV(RE*) = -20\*.25 + 30\*.35 + 90\*.40 = $41.5

*EMV(IN*) = -50\*.25 + 0\*.35 + 150\*.40 = $47.5

*EMV(NV*) = -250\*.25 + -20\*.35 + 300\*.40 = 50.5

Based on the max *EMV*, appropriate decision criterion for risk-indifference, *NV* is the best investment alternative.

1. *Suppose the decision maker us risk-averse whose utility for money is U(X) =* *where X is a sum of money and U(X) is his utility for this sum. Does the utility function imply decreasing, constant or decreasing risk-averse behavior?*

To answer this question we need to see if the utility function implies increasing, constant of decreasing risk premiums as the starting monetary position (‘wealth’) increases. Let’s define the convenient lottery of $0 and $1000 with even probabilities. We will calculate the risk premium for two arbitrary starting positions: $0 and $1000 and see what happens to the risk premium. If the starting position is $0, the lottery will result in $0 or $1000. *EMV* = $500. Certainty equivalent is calculated by answering the question: what is the sure amount whose utility is equal to the expected utility of the lottery? Expected utility of the lottery is .5\*(√ 0 + √1000) = 15.81, thus the sure value whose utility equals 15.81 is 15.812 = $250 which leads to a risk premium of *RP = EMV – CE* = 500 – 250 = $250. Assume now the starting point is now $1000 thus the lottery will end in $1000 or $2000 with expected utility of 0.5\*(√ 1000 + √2000) = 38.17 which corresponds to a *CE* of 38.172 = $1457.11 since the *EMV* of the lottery is $1,500, the risk premium is $42.89. This analysis shows that as the individual becomes richer his risk-aversion decreases.

1. *The decision maker’s utility for money is negative exponential with R = $5,000,000. Which investment is the best?*

We can calculate the *CE* of each investment and choose the best. (All $ amounts are in thousands)

*CE(RE)* = -5000 [ln(.251*e*20/5000 + .35*e*-30/5000 + *.*40*e*-90/5000)] = 41.31

*CE(IN)* = -5000 [ln(*.*25*e*50/5000 + .35*e*-0 + *.*40 *e*-150/5000)] = 46.76

*CE(NV)* = -5000 [ln(*.*25*e2*50/5000 + .35*e*20/5000 + *.*40 *e*-300/5000)] = 45.57

Thus the Index fund is the best investment. Notice that this investor has a fairly large capacity for risk. *R* = $5M, means he is willing to risk $2.5M for an even chance for a gain of $5M and thus a fairly risky investment is chosen.

1. *Assume the decision maker now has a logarithmic utility function with A= $10M. Which investment is best?*

*CE(RE*) = [(10,000-20).25(10,000+30).35(10,000+90).40] – 10,000 = 41.40

*CE(IN)* = [(10,000-50).25(10,000+0).35(10,000+150).40] – 10,000 = 47.13

*CE(NV)* = [(10,000-250).25(10,000-20).35(10,000+300).40] – 10,000 = 48.05

New venture is the selected investment for this investor.

Further exercise: Verify the choices made in parts c) and d) by calculating the expected utility of each investment.

1. As you will see this convention simplifies the process of assigning utilities to outcomes. [↑](#footnote-ref-1)