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Stress tensor from the trace anomaly in Reissner-Nordström spacetimes

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The effective action associated with the trace anomaly provides a general algorithm for approximating the expectation value of the stress tensor of conformal matter fields in arbitrary curved spacetimes. In static, spherically symmetric spacetimes, the algorithm involves solving a fourth order linear differential equation in the radial coordinate r for the two scalar auxiliary fields appearing in the anomaly action, and its corresponding stress tensor. By appropriate choice of the homogeneous solutions of the auxiliary field equations, we show that it is possible to obtain finite stress tensors on all Reissner-Nordström event horizons, including the extreme $Q = M$ case. We compare these finite results to previous analytic approximation methods, which yield invariably an infinite stress energy on charged black hole horizons, as well as with detailed numerical calculations that indicate the contrary. The approximation scheme based on the auxiliary field effective action reproduces all physically allowed behaviors of the quantum stress tensor, in a variety of quantum states, for fields of any spin, in the vicinity of the entire family ($0 \leq Q \leq M$) of RN horizons.

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I. INTRODUCTION

The evaluation of the energy-momentum-stress tensor $T^a{}_b$ of quantum matter in curved spacetimes is important for understanding the possible backreaction effects of matter on the large scale geometry of spacetime. Quantitative control of the stress tensor is needed especially in black hole and cosmological spacetimes with event horizons, where general considerations indicate that vacuum polarization and particle creation may lead to significant quantum effects which are cumulative with time. Such secular, macroscopic effects of quantum matter on the geometry of spacetime provide the intriguing possibility of observable consequences of quantum gravity at accessible energy scales, far below the Planck scale.

In the direct method of computing the quantum expectation value of the stress tensor, the first step is to solve the appropriate matter field equations, for a complete set of normal modes. With these solutions in hand, the Fock space of the quantum theory is constructed, a specific ''vacuum'' state $|\Psi\rangle$ in the Fock space chosen, and the expectation value $\langle\Psi|T^a{}_b|\Psi\rangle$ evaluated in the selected state, component by component, as a sum over the normal modes of the field. Since $T^a{}_b$ is a dimension four operator in four spacetime dimensions, the mode sum for its expectation value is quartically divergent. Hence a delicate regularization procedure, such as point splitting must be employed in order to identify and remove the short distance divergences in the mode sum, absorbing them into appropriate counterterms up to dimension four in the gravitational effective action [1]. Only after this regularization and subtraction procedure is performed can finite results for the renormalized $\langle T^a{}_b\rangle$ of physical interest be extracted.

Since the wave equation, mode functions, and stress tensors are different for fields with different spin, this procedure must be carried out independently for each quantum field of interest. Likewise, if one wishes to consider different quantum states in the Fock space, with different boundary conditions on the mode functions, the calculation must be repeated for each state. Because of the intricacy of the subtraction procedure, together with the numerical solution of the mode equations, which is usually required, it is often difficult to anticipate the general physical features of the result. Finally, if the geometry is modified, or allowed to respond dynamically, the entire calculation of $\langle T^a{}_b\rangle$ would have to be repeated for each new geometry and/or at each new time step. This direct method of calculating $\langle T^a{}_b\rangle$ is thus both time and computation intensive, and has limited the number of results for the stress tensor in fixed backgrounds to only a handful of special cases, rendering the consideration of dynamical black hole spacetimes varying with time in response to $\langle T^a{}_b\rangle$ prohibitively difficult, even in the case of exact spherical symmetry.

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Because of the difficulties involved in direct evaluation methods, considerable interest attaches to developing general and reliable algorithms for approximating $\langle T^a{}_b \rangle$ in general curved spacetimes. An approximate method that can capture the secular, macroscopic quantum effects on the geometry would be particularly interesting for applications to both cosmological and black hole spacetimes.

In a previous article a general approximation scheme based on the effective action and stress tensor obtained from the conformal or trace anomaly was introduced [2]. Although the effective action associated with the trace anomaly is not unique, as it is defined only up to arbitrary conformally invariant terms, a minimal generally covariant action can be found by direct integration of the anomaly [3,4]. The logarithmic scaling behavior of the effective action associated with the anomaly separates it from any of the other possible local or nonlocal Weyl invariant terms in the exact effective action, which do not share this logarithmic scaling property. The nonlocal anomaly action may be cast into a local form in a standard way by the introduction of one or more scalar auxiliary degree(s) of freedom [3–6]. Since there are two distinct cocycles in the nontrivial cohomology of the Weyl group in four dimensions [7], the most general representation of the anomaly action is in terms of two auxiliary scalar degrees of freedom, each satisfying fourth order linear differential equations of motion (2.8). These are two new scalar degrees of freedom in the low energy effective theory of gravity not present in the classical Einstein theory. Since the effective action expressed in terms of the auxiliary fields is a spacetime scalar, variation with respect to the metric yields a covariantly conserved stress tensor, which not only reproduces the trace anomaly but also yields nontrivial tracefree components as well.

In the auxiliary field approach, computation of the *quantum* expectation value $\langle T^a{}_b \rangle$ is reduced to the solution of linear, *classical* equations for the auxiliary fields, bypassing completely any summation over modes, and the regularization and renormalization that requires. Different states of the underlying quantum field(s) are associated with the choice of specific homogeneous solutions to the linear differential equations satisfied by the auxiliary fields. This allows for states obeying different boundary conditions on the horizon to be studied simultaneously. In addition, matter fields of every spin are treated in a unified manner, since the auxiliary field equations do not depend on the spin of the underlying quantum field. The stress tensor depends on the spin of the fields only through the known spin dependence of the two numerical coefficients appearing in the anomaly in Eqs. (2.3) below. Finally, the auxiliary field action and the stress tensor derived from it can be evaluated in principle in any spacetime, dynamical or not, without regard to special symmetries. Thus the scalar effective action of the anomaly furnishes a general classical algorithm for approximating the expectation value of the full stress tensor of quantum matter of any spin in an arbitrary curved spacetime.

The auxiliary fields are sensitive to macroscopic boundary conditions and the presence of causal horizons, so it is particularly interesting to apply the approximation algorithm based on them to spacetimes with event horizons, where quantum fluctuations are expected to play an important role, and comparison with existing numerical results is possible. Even when the quantum matter fields are not strictly massless, their fluctuations and stress tensor in the vicinity of an event horizon can exhibit conformal behavior. In particular, quantum states with diverging $\langle T^a{}_b \rangle$ on the Schwarzschild horizon, for which the backreaction on the classical geometry is significant, have precisely those diverging behaviors prescribed by the stress tensor derived from the effective action of the anomaly [2].

In Ref. [6] a study of the stress tensor obtained from the anomaly in Schwarzschild spacetime was undertaken. In Ref. [2] the general form of the stress tensor due to the conformal anomaly in an arbitrary spacetime was given and applied to a few special cases, such as Schwarzschild and de Sitter spacetimes. A detailed comparison of the two studies is given in Sec. IV.

In the present article we extend and develop the classical approximation technique of Ref. [2] for the quantum stress tensor based on the trace anomaly to static, spherically symmetric spacetimes, focusing specifically on electrically charged Reissner-Nordström (RN) black hole spacetimes, and states with regular stress tensors on the RN horizon. Since direct computations of the renormalized stress tensor expectation value $\langle T^a{}_b \rangle$ have been carried out in both Schwarzschild and Reissner-Nordström (RN) spacetimes for free fields of various spin [8–18], we will be able to compare the results of the new approximation scheme to these direct computations of $\langle T^a{}_b \rangle$. Since the exact one-loop effective action of quantum matter generally contains terms which are not determined by the anomaly, using it to compute the stress tensor expectation value is certainly an approximation, which will differ from the direct evaluation of $\langle T^a{}_b \rangle$ in terms of mode sums in general. It is therefore a nontrivial check of the approximation scheme if all of the allowed behaviors of the *traceless* parts of the exact $\langle T^a{}_b \rangle$ on event horizons can be reproduced by the auxiliary field method based on the trace anomaly.

Several approaches to approximating $\langle T^a{}_b \rangle$ have been discussed previously in the literature, developed with important special cases in mind, such as spherically symmetric geometries with a timelike Killing field [16,19–25]. These approximations are quite successful for regular states in the Schwarzschild geometry, but fail when compared to the numerical results for $\langle T^a{}_b \rangle$ in the charged RN spacetimes. Specifically, the previous approximations invariably yield a renormalized $\langle T^a{}_b \rangle$ which grows logarithmically without bound as the horizon of any charged RN black hole is approached. For the case of an extreme

Reissner-Nordström black hole there is an even stronger linear divergence. On the other hand direct numerical evaluation of $\langle T^a{}_b \rangle$ in the Hartle-Hawking-Israel [26] thermal state shows no evidence for any of these divergences [16–18]. We review the previous approximation methods and compare them both to the auxiliary field method and the direct evaluations of $\langle T^a{}_b \rangle$ in Sec. IV.

Our main purpose in this paper is to show that the auxiliary field effective action and stress tensor determined by the trace anomaly leads to a practical semianalytic approximation technique which allows for a finite $\langle T^a{}_b \rangle$ on the event horizons of all electrically charged black holes, including the extreme Reissner-Nordström (ERN) case of $Q = M$. Although the stress tensor diverges on the horizon for *generic* solutions of the auxiliary field equations, it is possible to adjust the homogeneous solutions of these linear equations to remove the divergences. The ERN case is particularly interesting, since its Hawking temperature vanishes and its degenerate horizon structure leads to potentially more severely divergent terms in the stress tensor. These leading divergent behaviors can be determined analytically by a power series expansion of the auxiliary fields in the local vicinity of the horizon, and explicitly cancelled, if desired.

The paper is organized as follows. In the next section we review the effective action and stress tensor of the trace anomaly in the auxiliary field form introduced in [2]. In Sec. III, we apply the general approximation algorithm to static, spherically symmetric spacetimes, reviewing briefly the uncharged Schwarzschild case, and then extending the analysis to the generic charged $Q < M$, RN cases, and the ERN, $Q = M$ case. We determine in each case the conditions on the series expansion coefficients of the auxiliary fields necessary for a regular $\langle T^a{}_b \rangle$ on the horizon. In Sec. IV we solve the regularity conditions and compare them to previous analytic approximation schemes, and direct numerical evaluations of $\langle T^a{}_b \rangle$. Section V contains our conclusions, while the appendix catalogs the complete list of solutions to the regularity conditions.

II. STRESS TENSOR FROM THE TRACE ANOMALY

Classical fields satisfying wave equations with zero mass, which are invariant under conformal transformations of the spacetime metric, $g_{ab} \to e^{2\sigma} g_{ab}$ have stress tensors with zero classical trace, $T^a{}_a = 0$. Because the corresponding quantum theory requires an ultraviolet (UV) regulator, classical conformal invariance cannot be maintained at the quantum level. The trace of the stress tensor is generally nonzero when $\hbar \neq 0$, and any UV regulator which preserves the covariant conservation of $T^a{}_b$, a necessary requirement of any theory respecting general coordinate invariance, yields an expectation value of the quantum stress tensor with a nonzero trace: $\langle T^a{}_a \rangle \neq 0$. This conformal or trace anomaly is therefore a general feature of quantum theory in gravitational fields, on the same footing as the chiral anomaly in QCD responsible for the experimentally measured decay of the π^0 meson into two photons [27].

In four spacetime dimensions the trace anomaly takes the general form [1,28],

$$\langle T^a{}_a \rangle = bF + b'\left(E - \frac{2}{3}\Box R\right) + b''\Box R + \sum_i \beta_i H_i. \quad (2.1)$$

In Eq. (2.1) we employ the notation,

$$E \equiv {}^*R_{abcd}{}^*R^{abcd} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2, \quad (2.2a)$$

and

$$F \equiv C_{abcd}C^{abcd} = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{R^2}{3}. \quad (2.2b)$$

with R_{abcd} the Riemann curvature tensor, ${}^*R_{abcd} = \frac{1}{2}\varepsilon_{abef}R^{ef}{}_{cd}$ its dual, and C_{abcd} the Weyl conformal tensor. The coefficients b, b', and b'' are dimensionless parameters proportional to $\hbar$. Additional terms denoted by the sum $\sum_i \beta_i H_i$ in (2.1) may also appear in the general form of the trace anomaly, if the massless field in question couples to additional long range gauge fields. Thus in the case of massless fermions coupled to a background gauge field, the invariant $H = \mathrm{tr}(F_{ab}F^{ab})$ appears in (2.1) with a coefficient β determined by the beta function of the relevant gauge coupling [29].

The form of (2.1) and coefficients b and b' do not depend on the state in which the expectation value of the stress tensor is computed. Instead they are determined only by the number of massless fields and their spin via

$$b = \frac{\hbar}{120(4\pi)^2}(N_S + 6N_F + 12N_V), \quad (2.3a)$$

$$b' = -\frac{\hbar}{360(4\pi)^2}(N_S + 11N_F + 62N_V), \quad (2.3b)$$

with N_S the number of spin 0 fields, N_F the number of spin $\frac{1}{2}$ Dirac fields, and N_V the number of spin 1 fields [1]. Henceforth we shall set $\hbar = 1$, although it should be remembered that any effect of the anomaly in which the b, b' and β_i coefficients appear is a one-loop quantum effect.

The trace anomaly determines the conformal variation of the one-loop effective action of the matter fields in a general curved background. A covariant, nonlocal form of this effective action was first given in Ref. [3]. One consequence of the effective action due to the anomaly is that the scalar or conformal part of the metric becomes dynamical, and its fluctuations provide a mechanism for the screening of the cosmological vacuum energy [30–32]. The stress tensor, canonical quantization of the conformal degree of freedom, and physical states of the quantum

conformal factor in the absence of the Einstein-Hilbert term were studied in Ref. [33].

That the nonlocal action of the anomaly could be rendered local by the introduction of scalar auxiliary field(s), was noted in Refs. [3–6]. Partial forms of the stress tensor due to this effective action were given in [6,7,33], with the authors of [6] initiating the study of the stress tensor obtained from the effective action of the anomaly with auxiliary fields as an approximation scheme in the Ricci flat case of Schwarzschild spacetime. The general, complete form of the stress tensor in terms of two auxiliary fields in an arbitrary curved spacetime was given in [2]. This auxiliary field effective action is of the form,

$$S_{\rm anom} = b' S^{(E)}_{\rm anom}[g;\varphi] + b S^{(F)}_{\rm anom}[g;\varphi,\psi], \tag{2.4}$$

with

$$\begin{aligned} S^{(E)}_{\rm anom}[g;\varphi] &\equiv \frac{1}{2}\int d^4x\sqrt{-g}\Big\{-(\Box\varphi)^2 + 2\Big(R^{ab} - \frac{R}{3}g^{ab}\Big)(\nabla_a\varphi)(\nabla_b\varphi) + \Big(E - \frac{2}{3}\Box R\Big)\varphi\Big\}, \\ S^{(F)}_{\rm anom}[g;\varphi,\psi] &\equiv \int d^4x\sqrt{-g}\Big\{-(\Box\varphi)(\Box\psi) + 2\Big(R^{ab} - \frac{R}{3}g^{ab}\Big)(\nabla_a\varphi)(\nabla_b\psi) + \frac{1}{2}F\varphi + \frac{1}{2}\Big(E - \frac{2}{3}\Box R\Big)\psi\Big\}, \end{aligned} \tag{2.5}$$

in terms of the two scalar auxiliary fields φ and ψ, corresponding to the two nontrivial cocycles of the Weyl group in four dimensions [7].

The effective action (2.4) and (2.5) is a spacetime scalar integral over local fields. Hence varying it with respect to the metric yields two covariantly conserved stress tensors E_{ab} and F_{ab}, bilinear in the scalar auxiliary fields. Explicitly, these are [2]

$$\begin{aligned} E_{ab} = {} & -2(\nabla_{(a}\varphi)(\nabla_{b)}\Box\varphi) + 2\nabla^c[(\nabla_c\varphi)(\nabla_a\nabla_b\varphi)] - \frac{2}{3}\nabla_a\nabla_b[(\nabla_c\varphi)(\nabla^c\varphi)] + \frac{2}{3}R_{ab}(\nabla_c\varphi)(\nabla^c\varphi) - 4R^c{}_{(a}(\nabla_{b)}\varphi)(\nabla_c\varphi) \\ & + \frac{2}{3}R(\nabla_a\varphi)(\nabla_b\varphi) + \frac{1}{6}g_{ab}\{-3(\Box\varphi)^2 + \Box[(\nabla_c\varphi)(\nabla^c\varphi)] + 2(3R^{cd} - Rg^{cd})(\nabla_c\varphi)(\nabla_d\varphi)\} - \frac{2}{3}\nabla_a\nabla_b\Box\varphi \\ & - 4C_a{}^c{}_b{}^d\nabla_c\nabla_d\varphi - 4R^c_{(a}\nabla_{b)}\nabla_c\varphi + \frac{8}{3}R_{ab}\Box\varphi + \frac{4}{3}R\nabla_a\nabla_b\varphi - \frac{2}{3}(\nabla_{(a}R)\nabla_{b)}\varphi \\ & + \frac{1}{3}g_{ab}\{2\Box^2\varphi + 6R^{cd}\nabla_c\nabla_d\varphi - 4R\Box\varphi + (\nabla^c R)\nabla_c\varphi\}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} F_{ab} = {} & -2(\nabla_{(a}\varphi)(\nabla_{b)}\Box\psi) - 2(\nabla_{(a}\psi)(\nabla_{b)}\Box\varphi) + 2\nabla^c[(\nabla_c\varphi)(\nabla_a\nabla_b\psi) + (\nabla_c\psi)(\nabla_a\nabla_b\varphi)] - \frac{4}{3}\nabla_a\nabla_b[(\nabla_c\varphi)(\nabla^c\psi)] \\ & + \frac{4}{3}R_{ab}(\nabla_c\varphi)(\nabla^c\psi) - 4R^c{}_{(a}[(\nabla_{b)}\varphi)(\nabla_c\psi) + (\nabla_{b)}\psi)(\nabla_c\varphi)] + \frac{4}{3}R(\nabla_{(a}\varphi)(\nabla_{b)}\psi) \\ & + \frac{1}{3}g_{ab}\{-3(\Box\varphi)(\Box\psi) + \Box[(\nabla_c\varphi)(\nabla^c\psi)] + 2(3R^{cd} - Rg^{cd})(\nabla_c\varphi)(\nabla_d\psi)\} - 4\nabla_c\nabla_d(C_{(a}{}^c{}_{b)}{}^d\varphi) - 2C_a{}^c{}_b{}^dR_{cd}\varphi \\ & - \frac{2}{3}\nabla_a\nabla_b\Box\psi - 4C_a{}^c{}_b{}^d\nabla_c\nabla_d\psi - 4R^c_{(a}(\nabla_{b)}\nabla_c\psi) + \frac{8}{3}R_{ab}\Box\psi + \frac{4}{3}R\nabla_a\nabla_b\psi - \frac{2}{3}(\nabla_{(a}R)\nabla_{b)}\psi \\ & + \frac{1}{3}g_{ab}\{2\Box^2\psi + 6R^{cd}\nabla_c\nabla_d\psi - 4R\Box\psi + (\nabla^c R)(\nabla_c\psi)\}. \end{aligned} \tag{2.7}$$

These tensors have the local geometrical traces,

$$E^a{}_a = 2\Delta_4\varphi = E - \frac{2}{3}\Box R, \tag{2.8a}$$

$$F^a{}_a = 2\Delta_4\psi = F = C_{abcd}C^{abcd}, \tag{2.8b}$$

where the latter half of these two equations follows from the independent Euler-Lagrange variation of (2.4) and (2.5) with respect to the two auxiliary scalar degrees of freedom, φ and ψ.

The fourth order scalar differential operator appearing in these expressions is [2,3,33]

$$\begin{aligned} \Delta_4 &\equiv \Box^2 + 2R^{ab}\nabla_a\nabla_b - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^a R)\nabla_a \\ &= \nabla_a\Big(\nabla^a\nabla^b + 2R^{ab} - \frac{2}{3}Rg^{ab}\Big)\nabla_b. \end{aligned} \tag{2.9}$$

By solving the fourth order linear Eqs. (2.8) determined by this Δ_4 for the two auxiliary fields, φ and ψ, and substituting the results into the stress tensors (2.6) and (2.7) we obtain a general approximation algorithm for $\langle T^\mu{}_\nu\rangle$ for conformal matter fields of any spin in an arbitrary curved spacetime. That is,

$$\langle T^{\mu}{}_{\nu}\rangle \simeq T^{\mu}{}_{\nu}[\varphi,\psi] = b' E^{\mu}{}_{\nu} + b F^{\mu}{}_{\nu}. \tag{2.10}$$

is an approximation to the exact stress tensor expectation value. Since the dependence of the stress tensor $T^{\mu}{}_{\nu}[\varphi,\psi]$ on the spin of the underlying quantum matter fields arises purely through the numerical coefficients b and b' through (2.3), and β_i, there are no new equations to be solved for quantum fields of different spin.

The freedom to add homogeneous solutions of (2.8) to any given inhomogeneous solution allows the tracefree components of the stress tensor (2.10) to be changed without altering its trace. This corresponds to the freedom to change the boundary conditions and the state of the underlying quantum field theory without changing its state independent trace anomaly. As shown in [2] the auxiliary fields and traceless terms in the stress tensor (2.10) generally diverge on event horizons, which provides a coordinate invariant meaning to large quantum backreaction effects on horizons. These state-dependent divergences can be analyzed and removed by specifying boundary conditions for the auxiliary fields on the horizon. One then has an approximation scheme for the expectation value $\langle T^{\mu}{}_{\nu}\rangle$ in regular states as well.

In order to characterize the nature of the approximation (2.10), we recall the general decomposition of the exact quantum effective action into three parts,

$$S_{\text{exact}} = S_{\text{local}} + S_{\text{inv}} + S_{\text{anom}}, \tag{2.11}$$

according to its transformation properties under global Weyl transformations [7]. The local action S_{local} can be expressed purely in terms of local contractions of the Riemann curvature tensor and its derivatives. In addition to the classical Einstein-Hilbert action S_{local} consists of an infinite series of higher dimension curvature invariants multiplied by increasing powers of an inverse mass scale. These terms, consistent with a general effective field theory analysis of gravity, give higher order geometric contributions to the stress tensor, which remain bounded and small for small curvatures. The two remaining terms in (2.11) are generally nonlocal. Any nonlocal terms involving a nonzero *fixed* mass parameter can be expanded in a series of higher derivative local terms multiplied by powers of the inverse mass and regrouped into S_{local}. Hence we need consider only those nonlocal terms which are *not* associated with any fixed mass or length scale in the remaining terms of S_{exact}. These must be either strictly Weyl invariant, denoted by S_{inv} in (2.11), or break local Weyl invariance, yet without introducing any explicit mass or length scale. Up to possible surface terms, these are just the geometric terms required by the trace anomaly (2.1). The associated terms in S_{anom} scale logarithmically with distance or energy, and are composed of the two distinct cocycles of the Weyl group, given by (2.4).

When the background spacetime is conformally flat or approximately so, the Weyl invariant action S_{inv} may be neglected, since it vanishes in the conformally related flat spacetime, in the usual Poincaré invariant vacuum state. In that case we expect S_{anom} to become a good approximation to the nonlocal terms in the exact effective action (2.11), and the corresponding stress tensor (2.10) to become a good approximation to the exact quantum stress tensor, up to well-known local terms. Thus the approximation (2.10) amounts to the neglect of S_{inv}, or more precisely, those parts of S_{inv} which cannot be expressed in terms of local terms or parametrized by homogeneous solutions to the auxiliary field Eqs. (2.8).

Because of the conformal behavior of fields near an event horizon, where the effects of mass terms become subdominant, one might expect the leading behavior of the stress tensor (2.10) also to match that of the exact $\langle T^{a}{}_{b}\rangle$ in the vicinity of the horizon. In [2] we tested this hypothesis in Schwarzschild and de Sitter spacetimes, finding that the freedom to choose homogeneous solutions to the auxiliary field equations (2.8) allows for all possible allowed behaviors of the exact stress tensor near the horizon. Indeed for states with diverging stress tensor on the horizon, the anomalous stress tensor (2.10) gives the correct leading and subleading behaviors of the exact $\langle T^{a}{}_{b}\rangle$.

When attention is restricted to states with regular behavior on the horizon, which from the point of view of conformal invariance are *subleading* with respect to the divergent terms, then the stress tensor (2.10) can often be adjusted to give the exact finite value of $\langle T^{a}{}_{b}\rangle$ on the horizon as well. However, the global fit of (2.10) is in only fair quantitative agreement with the numerically computed expectation value $\langle T^{a}{}_{b}\rangle$ far from the horizon for the Hartle-Hawking-Israel state in Schwarzschild spacetime. In regular states the neglected terms in S_{exact}, which remain bounded at the horizon, are comparable in magnitude to subleading terms of S_{anom}. Hence neglect of S_{inv} is expected to yield a poorer global approximation to the stress tensor of such regular states even when it is possible to match the exact behavior of $\langle T^{a}{}_{b}\rangle$ at the horizon.

The Schwarzschild and de Sitter cases considered in [2] are special in that the auxiliary field Eqs. (2.8) may be solved analytically. However, the approximation (2.10) does not require this, and in the following we extend the auxiliary field method to general static, spherically symmetric geometries, focusing specifically on the RN family of charged black holes. These black hole spacetimes provide an interesting test bed for the auxiliary field stress tensor, whose qualitative and quantitative features may be compared with both previous approximation methods and direct numerical evaluation of $\langle T^{a}{}_{b}\rangle$.

III. REGULAR STRESS TENSORS IN REISSNER-NORDSTRÖM SPACETIMES

The effective action and stress tensor of the auxiliary fields (2.10) is defined in any spacetime, regardless of special symmetries. However it becomes particularly simple as a method of approximating the expectation value of

the quantum stress tensor in spacetimes with a high degree of symmetry, such as spherical symmetry. The line element for a general static, spherically symmetric spacetime can be expressed in terms of two functions of the radius in the form,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2. \tag{3.1}$$

Assuming that the state in which we evaluate the stress tensor is also spherically symmetric and stationary in time, we may make a static, spherically symmetric ansatz for the auxiliary fields as well, i.e.

$$\varphi = \varphi(r) \tag{3.2a}$$

$$\psi = \psi(r), \tag{3.2b}$$

so that Eqs. (2.8) become ordinary differential equations in r. In some cases it is possible also to add terms with linear time dependence to the auxiliary fields, i.e. $\varphi = \varphi(r) + \alpha t$ and $\psi = \psi(r) + \alpha' t$ in order to allow for the possibility of nonvanishing $T^r{}_t$ flux components which are also independent of the Killing time t. Higher powers of t or more complicated time dependence in the auxiliary fields lead to nonstationary stress tensors.

For a general static, spherically symmetric spacetime, with the fields in a spherically symmetric quantum state, the stress tensor is given by its three independent diagonal components,

$$T^t{}_t = -\rho(r) \tag{3.3}$$

$$T^r{}_r = p(r) \tag{3.4}$$

$$T^\theta{}_\theta = p_\perp(r), \tag{3.5}$$

together with a possible nonzero off-diagonal flux component $T^r{}_t$. These components obey the covariant conservation conditions,

$$\nabla_a T^a{}_r = \frac{dp}{dr} + \frac{1}{2f}\frac{df}{dr}(p+\rho) + \frac{2}{r}(p - p_\perp) = 0 \tag{3.6}$$

and

$$\nabla_a T^a{}_t = \frac{1}{\sqrt{-g}}\frac{d}{dr}(\sqrt{-g}T^r{}_t) = 0. \tag{3.7}$$

Equation (3.7) can be integrated immediately to obtain

$$T^r{}_t = -\frac{L}{4\pi r^2}\sqrt{\frac{h}{f}}, \tag{3.8}$$

with the integration constant L the luminosity of a localized source.

The approximation of (2.10) can be applied to arbitrary spacetimes, spherically symmetric or not, with or without an horizon. For definiteness we restrict our attention in this paper to the family of Reissner-Nordström spacetimes with the equal metric functions,

$$f(r) = h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{3.9}$$

depending upon M, the mass and Q, the electric charge. We then can distinguish three cases:

(i) $Q = 0$, Schwarzschild spacetime;
(ii) $0 < Q < M$, generic Reissner-Nordström (RN) spacetime;
(iii) $Q = M$, extreme Reissner-Nordström (ERN) spacetime.

We shall discuss each of these cases in detail separately.

Since (3.9) is quadratic in $1/r$ there are two values of r at which $f(r)$ vanishes. These are

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{3.10}$$

When $Q = 0$, $r_+ = 2M$ is the usual Schwarzschild horizon. In the ERN case of $Q = M$, the two values $r_\pm$ coincide and the horizon becomes degenerate. The character of the spacetime, the solutions to the auxiliary field equations and the corresponding stress tensors derived from them are quite different in each of the three cases. Henceforth we reserve the designation RN for the generic, charged black hole solution of case (ii).

For conformal field theories, the trace of the stress tensor is given purely by the trace anomaly. This provides us with another relation for the diagonal components of the tensor, namely

$$T^a{}_a = -\rho + p + 2p_\perp = b'E + bF. \tag{3.11}$$

Furthermore, defining

$$\Theta(r) \equiv p_\perp - \frac{T^a{}_a}{4}, \tag{3.12}$$

and integrating (3.6) with (3.11) gives [34]

$$p(r) = \frac{1}{r^2 f}\int_{r_+}^{r} dr(2rf - r^2 f')\Theta + \frac{1}{4r^2 f}\int_{r_+}^{r} dr\frac{d(r^2 f)}{dr}T^a{}_a + \frac{C - L}{4\pi r^2 f}, \tag{3.13}$$

where $C - L$ denotes an overall constant of integration. From Eq. (3.8) with $h(r) = f(r)$, the off-diagonal flux component is

$$T^r{}_t = -\frac{L}{4\pi r^2}. \tag{3.14}$$

The only unknown function in these expressions is $\Theta(r)$. Thus it is sufficient to compute $\Theta(r)$ and determine the integration constants C and L from the auxiliary field stress tensor to determine subsequently all the nonzero components of the stress tensor, $p(r)$, $p_\perp(r)$, $\rho(r)$, and $T^r{}_t$ in a stationary state in static coordinates (3.1) from Eqs. (3.11), (3.12), (3.13), and (3.14) respectively.

Since the curvature invariants in the trace (3.11) remain finite as the horizon at $r = r_+$ is approached, it is clear

from (3.12) that $p_\perp$ is finite on the horizon, provided $\Theta(r)$ remains finite there. Moreover if $f(r)$ possesses an isolated simple zero at $r = r_+$, vanishing linearly in[1]

$$s = (r - r_+)/r_+ \tag{3.15}$$

as $r \to r_+$, so that the event horizon is nondegenerate, then (3.13) shows that a divergent term in the other components of the stationary stress tensor can arise only if $C - L \neq 0$. Hence $\Theta(r)$ remaining finite as $r \to r_+$, and $C = L$, are necessary and sufficient conditions for finiteness of all components of $\langle T^a{}_b \rangle$ in the static Killing frame of (3.1) for the nondegenerate Reissner-Nordström horizons ($0 \leq Q < M$).

On the other hand if one requires finiteness of the stress tensor in the frame of a freely falling observer then [34]

$$\frac{|T_{uu}|}{f^2} = \frac{1}{4f^2}\left|(\rho + p)f + \frac{L}{2\pi r^2}\right| < \infty. \tag{3.16}$$

If $\Theta(r)$ is regular at $r = r_+$ and can be expanded in a Taylor series in s near $s = 0$, then it is easy to see that the condition (3.16) is satisfied automatically if $C = 0$. In the case of the past horizon one should demand finiteness of $|T_{vv}|/f^2$ instead of (3.16). This forces the integration constant to be $C = 2L$ instead. Thus regularity on both the past and future horizon requires $\Theta(r_+) < \infty$ and $C = L = 0$. This is the only case for which finiteness of $T^a{}_b$ in *both* the static Killing frame (3.1) and the frame of freely falling observers (3.16) can be maintained. In particular, although $\Theta(r_+)$ finite and $C = 0$ are sufficient to guarantee that (3.16) is satisfied, any $L \neq 0$ leads to divergent behavior of $p(r)$ and hence $\rho(r)$ from (3.13). Although the auxiliary field method allows for $L \neq 0$, we shall focus in this paper on stationary states with completely finite stress tensors, for which $C = L = 0$. In this case the regularity condition (3.16) is equivalent to the condition,

$$|T^{uu}| = \frac{1}{f}|\rho + p| < \infty, \qquad C = L = 0. \tag{3.17}$$

In some cases, i.e. those with logarithmic terms in the auxiliary fields, $\Theta(r)$ is not analytic at $r = r_+$ and cannot be expanded in a Taylor series there. In those cases, the finiteness condition (3.16) will give additional conditions on the behavior of the auxiliary fields on the horizon. Equation (3.17) also shows that finiteness of T^{uu} on the horizon requires $T^r{}_r = p = -\rho = T^t{}_t$ there.

Before restricting our attention to the static, completely regular class of stress tensors, let us emphasize that this is not the generic case, as a different choice of the free integration constants C and L would lead to different physical behavior on the horizon. It is well known that the coordinate singularity of the metric (3.1) at the horizon $r = r_+$ where $f = h$ vanishes may be removed by a (singular) coordinate transformation, producing the complex analytic extension of the Reissner-Nordström geometry [35]. Although any regular transformation of coordinates is allowed by the equivalence principle, and cannot lead to physical effects, *singular* coordinate transformations, like singular gauge transformations in gauge theory, must be treated with some care. New topological configurations such as monopoles or vortices are associated with such singular gauge transformations. Thus, although the complex analytic extension of a black hole spacetime may seem quite natural mathematically, analytic continuation actually involves a physical assumption, namely, that there are no stress tensor sources to the Einstein equations localized on or near the horizon. Because of the hyperbolic nature of Einstein's equations such stress sources with effects transmitted along a null surface are perfectly allowable, even classically. When the expectation value of the stress tensor of quantum fields is considered, with its sensitivity to the wavelike, nonlocal coherence effects of quantum matter, the assumption of analyticity on the horizon is not at all automatic, and is not required by any general principles of quantum theory.

The effective action of the conformal anomaly, and its associated auxiliary fields indicate that nonregularity of the stress tensor on the horizon is to be expected in the generic case as well [2]. A tuning of the integration constants of the solutions of the linear equations (2.8) for φ and ψ is necessary to prevent $\Theta(r)$ from diverging as $r \to r_+$, with the generic behavior of all the stress tensor components near the horizon being proportional to f^{-2} as $f \to 0$ in the Schwarzschild case. Since the auxiliary fields are spacetime scalars, it is clear that this behavior is in no contradiction to the equivalence principle. In fact, the divergences have a perfectly coordinate invariant origin in terms of the homogeneous solutions to Eqs. (2.8),

$$\varphi_h = \psi_h = \ln(-K^a K_a) = \ln f(r) \tag{3.18}$$

where $K = \frac{\partial}{\partial t}$ is the Killing field of the static Schwarzschild or Reissner-Nordström geometries, timelike for $r > r_+$. This defines the rest frame of the configuration, which is independent of coordinate redefinitions. The allowed divergences in the stress tensor on the horizon as $f(r) \to 0$ and $\varphi, \psi \to \pm\infty$ are related therefore to the behavior of this Killing invariant of the global geometry becoming null. The state of the quantum matter fields, specified on a complete Cauchy surface of the spacetime is necessarily defined in a nonlocal way, and hence expectation values of $T^a{}_b$ can be sensitive to the divergences of (3.18) at this null surface, notwithstanding the finiteness of the local Riemann curvature at $r = r_+$.

By choosing suitably restricted solutions of the auxiliary field equations (2.8) it is possible to cancel the $\ln f$ behavior in φ and ψ, thereby guaranteeing that $\Theta(r_+)$ is finite. When the electric charge satisfies $0 < Q < M$, i.e. excluding the uncharged Schwarzschild and maximally charged

[1]Note that this is a slightly different definition of s than used in Refs. [17,18] where $s \equiv (r - r_+)/M$.

ERN cases, the stress tensor will have subleading $\ln s$ and $\ln^2 s$ divergences as $s \to 0$ as well. Then $\Theta(r)$ is not expandable as a Taylor series around $r = r_+$ and the logarithmic terms do not drop out entirely from the condition (3.16). This leads to additional conditions on the coefficients of the expansion around $s = 0$ for a fully regular stress tensor.

A. Schwarzschild spacetime

In the Schwarzschild case, $f(r) = 1 - 2M/r$, and the fourth order linear Eqs. (2.8) can be integrated explicitly for auxiliary fields which are functions only of r. The result is [2,6]

$$\left.\frac{d\varphi}{dr}\right|_S = \frac{q-2}{6M}\left(\frac{r}{2M}+1+\frac{2M}{r}\right)\ln\left(1-\frac{2M}{r}\right) - \frac{q}{6r}\left[\frac{4M}{r-2M}\ln\left(\frac{r}{2M}\right)+\frac{r}{2M}+3\right] - \frac{1}{3M} - \frac{1}{r} + \frac{2Mc_H}{r(r-2M)} + \frac{c_\infty}{2M}\left(\frac{r}{2M}+1+\frac{2M}{r}\right) \tag{3.19}$$

in terms of the three dimensionless constants of integration, c_H, c_∞, and q. A fourth integration constant would be introduced by integrating (3.19) once further, but as the stress tensor in the Schwarzschild case depends only upon derivatives of φ, a constant shift in φ plays no role in this case.

The role of the three integration constants appearing in (3.19) is best exposed by examining the limits,[2]

$$\left.\frac{d\varphi}{dr}\right|_S \to \frac{c_H}{r-2M} + \frac{q-2}{2M}\ln\left(\frac{r}{2M}-1\right) + \frac{1}{2M}\left(3c_\infty - c_H - q - \frac{5}{3}\right) + \ldots, \qquad r \to 2M, \tag{3.20a}$$

$$\left.\frac{d\varphi}{dr}\right|_S \to \frac{c_\infty r}{4M^2} + \frac{2c_\infty - q}{4M} + \frac{c_\infty}{r} - \frac{2M}{3r^2}q\ln\left(\frac{r}{2M}\right) + \frac{2M}{r^2}\left[c_H - \frac{7}{18}(q-2)\right] + \ldots, \qquad r \to \infty. \tag{3.20b}$$

Hence c_H controls the leading behavior as r approaches the horizon, corresponding to the homogeneous solution of (3.18). It is this leading behavior that gives rise to the generic f^{-2} behavior of the stress tensor as $r \to r_+$. The second integration constant c_∞ controls the leading behavior of $\varphi(r)$ as $r \to \infty$, which is the same as in flat space. Nonzero values of c_∞ correspond to nontrivial boundary conditions at some large but finite volume, such as may be appropriate in the Casimir effect, or if the black hole is enclosed in a box. The constant q is the topological charge of the auxiliary field configuration, associated with the conserved current generated by the Noether symmetry of the effective action (2.4), $\psi \to \psi + \text{const}$ [2]. It is responsible for the $\ln r$ terms in (3.20b) and the corresponding stress tensor (2.6).

To the general spherically symmetric static solution (3.19) we may add also a term linear in t, i.e. we may replace $\varphi(r)$ by

$$\varphi(r,t) = \varphi(r) + \frac{\eta}{2M}t, \tag{3.21}$$

with η an additional free constant of integration. Linear time dependence in the auxiliary fields is the only allowed time dependence that leads to a time-independent stress energy, and this only in the Ricci flat Schwarzschild case. The solution for $\psi = \psi(r,t)$ is of the same form as (3.19) and (3.21) with four new integration constants, d_H, d_∞, q' and η' replacing c_H, c_∞, q, and η in $\varphi(r,t)$. Adding terms with any higher powers of t or more complicated t dependence produces a time dependent stress-energy tensor.

The stress energy diverges on the horizon in an entire family of states for generic values of the eight auxiliary field parameters $(c_H, q, c_\infty, \eta; d_H, q', d_\infty, \eta')$. Hence in the general allowed parameter space of spherically symmetric macroscopic states, horizon divergences of the stress energy are quite generic, and not restricted to the Boulware state [36]. In addition to the leading s^{-2} behavior, there are subleading s^{-1}, $\ln^2 s$ and $\ln s$ divergences in general. It turns out that only three of these four are independent, and the three conditions,

$$-(b'c_H + 2bd_H)c_H + \eta(b'\eta + 2b\eta') = 0 \qquad (s^{-2} \text{ in } \Theta) \tag{3.22a}$$

$$(q-2)[b'(q-2) + 2b(q'-2)] = 0 \qquad (\ln^2 s \text{ in } \Theta) \tag{3.22b}$$

$$b[(q-2)(18d_\infty - 30d_H - 40) + (q'-2)(18c_\infty - 30c_H - 40)] + b'(q-2)(18c_\infty - 30c_H - 40) = 0 \qquad (\ln s \text{ in } \Theta) \tag{3.22c}$$

[2]The last term of Eq. (3.20a) corrects a sign error in Eq. 5.9a of Ref. [2].

are all that are required to remove the divergent behaviors in Θ, indicated in parentheses as $s \to 0$, including a possible s^{-1} divergence at the horizon. The first of these conditions, (3.22a) corrects a sign error in Eq. (5.14b) of Ref. [2] (where the notations p, p' were used in place of the present η, η' for the parameters of the linear time dependent terms in φ, ψ respectively). The last two conditions of (3.22) remove the leading logarithmic behavior of $\Theta(r)$ at the horizon.

The luminosity in the general stationary state described by the auxiliary fields is

$$L = \frac{\pi}{M^2}[b' q \eta + b(q\eta' + q'\eta)]. \tag{3.23}$$

This shows that if we wish to obtain a state with $L \neq 0$ then one or both of the linear time dependences of the auxiliary fields η, η' must be nonzero.

The regularity condition (3.16) gives in addition,

$$b\left(\frac{2-q'}{3} + 2c_H + 2d_H\right) + b'\left(\frac{2-q}{3} + 2c_H\right) = b' q \eta + b(q\eta' + q'\eta) \qquad (C = 0) \tag{3.24a}$$

$$b[c_H(q'-2) + d_H(q-2)] + b' c_H (q-2) = 0 \qquad \left(\ln s \text{ in } \frac{T_{uu}}{f^2}\right) \tag{3.24b}$$

The first of these two conditions is also equivalent to the condition that a possible s^{-2} leading behavior in T_{uu}/f^2 vanish. Notice that the right-hand side of this condition is proportional to L from (3.23). The second condition (3.24b) eliminates a possible subleading logarithmic divergence in T_{uu}/f^2 on the horizon. With (3.22) and (3.24) satisfied there is no additional condition required for a possible $\ln^2 s$ term in T_{uu}/f^2 to vanish on the horizon as well.

If we restrict out interest to strictly static, regular states with $C = L = 0$ in (3.13) and (3.14), then (3.23) and hence (3.24a) vanishes. Also inspection of (3.24b) shows that the choice $q = q' = 2$ satisfies this condition as well as (3.22b) and (3.22c). This illustrates the general property that when all logarithmic terms are taken to vanish on the horizon, the conditions that $\Theta(r_+)$ be finite and $C = L = 0$ are sufficient to yield a fully regular static stress tensor, and the number of independent conditions is reduced.

B. Generic charged RN spacetimes

The generic $Q > 0$ charged Reissner-Nordström spacetimes are not Ricci flat and an analytic solution of the fourth order Eqs. (2.8) in closed form no longer appears possible. In addition, another difference from the Schwarzschild case, stemming from the same nonvanishing of the Ricci tensor when $Q > 0$, is that the linear time dependence (3.21) in the auxiliary field φ now produces time dependence in the stress tensor (2.7), and hence is disallowed for a static state. Linear time dependence in ψ is still allowed, and parametrizes a nonzero flux, $L \neq 0$. For strictly static states with $L = 0$ we must set $\eta = \eta' = 0$. Hence we apparently have one fewer integration constant than in the Schwarzschild case. However, the non-Ricci flatness means also that a possible constant term in φ which drops out of the stress tensor (2.10) in the case $R_{ab} = 0$ now survives as a nontrivial free parameter in the general RN case. Hence we have seven remaining integration constants in all for completely regular static stress tensors in the charged RN case, exactly the same number as in the Schwarzschild case after (3.23) is set to zero.

Since the fourth order differential operator, Δ_4 involves a total derivative, cf. Eq. (2.9), Eqs. (2.8) can be integrated once, to obtain the *second* order equation for $\varphi'(r)$,

$$f r^2 \frac{d}{dr}\left[\frac{1}{r^2}\frac{d}{dr}(f r^2 \varphi')\right] - 2Q^2 \frac{f}{r^2}\varphi' = e_0 + \frac{1}{2}\int_{r_+}^{r} r^2 dr E, \tag{3.25}$$

where e_0 is an integration constant. Here we have used the facts that

$$R^r{}_r = -\frac{Q^2}{r^4} \tag{3.26}$$

and $R = 0$ in a general RN spacetime.

A power series representation of the general solution of the second order Eq. (3.25) for φ' in powers of $s = (r - r_+)/r_+$ is easily derived. Since the right side of (3.25) is regular at $s = 0$, it can be expressed as a Taylor series in the form,

$$e_0 + \frac{1}{2}\int_{r_+}^{r} r^2 dr E = \sum_{n=0} e_n s^n. \tag{3.27}$$

To find the leading behavior of φ' near $s = 0$ let $\varphi' \sim s^\gamma$ for some γ, and thereby obtain from (3.25) and (3.27),

$$\begin{aligned}&\epsilon^2 \gamma(\gamma+1) s^\gamma + 2\epsilon(1-\epsilon)\gamma(\gamma+2) s^{\gamma+1} \\ &\quad + \{\gamma(\gamma+3)(1-4\epsilon) + \epsilon^2(3\gamma^2 + 9\gamma - 2)\} s^{\gamma+2} \\ &\quad + \mathcal{O}(s^{\gamma+3}) = \sum_{n=0} e_n s^n,\end{aligned} \tag{3.28}$$

where

$$\epsilon \equiv \frac{r_+ - r_-}{r_+} = \frac{2\sqrt{M^2 - Q^2}}{M + \sqrt{M^2 - Q^2}} \tag{3.29}$$

in the general case. From (3.28) we observe that $\gamma = -1$ gives the most singular behavior allowed for φ' as $s \to 0$ for general $e_0 \neq 0$, and $r_+ > r_-$. Note that $\varphi' \sim s^{-1}$

agrees also with the leading behavior in (3.20a) for the exact solution in the uncharged Schwarzschild case.

Because the singular s^{-1} behavior is allowed for φ' for all $0 \leq Q < M$, φ has at most logarithmically singular behavior near the RN horizon, and its general series expansion is of the form,

$$\varphi(r) = \sum_{n=0} a_n s^n + \sum_{n=0} \ell_n s^n \ln s. \tag{3.30}$$

Note that the general logarithmic behavior near $s = 0$ is that expected from (3.18) on geometrical grounds, since $e^{-\varphi_h}$ with $\varphi_h = \ln f$ is proportional to the conformal transformation needed to bring the RN metric to the ultrastatic optical metric, and remove the singularity caused by the Killing field changing character from timelike to null.

Substitution of the expansion (3.30) into the differential Eq. (3.25) gives recursion relations for the coefficients $\{a_n, \ell_n\}$. The set of four coefficients, $(a_0, a_1, \ell_0, \ell_1)$ are free integration constants for φ, parametrizing the general solution of the fourth order equation, with all higher order $a_{n>1}$ and $b_{n>1}$ determined by the recursion relations in terms of these four integration constants. In the Schwarzschild limit, $\ell_0 \to c_H$, $\ell_1 \to q - 2$, $a_1 \to 3c_\infty - 2q - c_H + \frac{1}{3}$ respectively. The constant e_0 is also determined in terms of ℓ_0 and ϵ by (3.25) or (3.28) to be

$$e_0 = -2\epsilon\ell_0 \frac{Q^2}{r_+^2} = -2\epsilon(1-\epsilon)\ell_0. \tag{3.31}$$

In like manner substitution of the series expansion,

$$\psi(r) = \sum_{n=0} c_n s^n + \sum_{n=0} \lambda_n s^n \ln s, \tag{3.32}$$

into the equation for ψ shows that $(c_0, c_1, \lambda_0, \lambda_1)$ are the four free integration constants parametrizing the general solution of (2.8b). However, since a constant shift in ψ is an exact symmetry of the action (2.4), c_0 does not appear in the stress tensor (2.10), and we are left with the seven effective free integration constants, $(a_0, a_1, \ell_0, \ell_1; c_1, \lambda_0, \lambda_1)$ in total.

The general form of the divergences of the anomalous stress tensor at the RN event horizon may be found from the power series expansions of the auxiliary fields there. In the $T^\theta{}_\theta$ component there are s^{-2}, s^{-1}, $\ln^2 s$ and $\ln s$ divergences. The leading s^{-2} behavior in all components of $T^a{}_b$ in coordinates (3.1) may be understood from the approximate conformal symmetry which applies near the horizon and the conformal weight of the stress tensor. Namely, since the near horizon geometry is conformally flat, with $e^{-\varphi_h \pm t/r_+}$ the conformal transformation to locally flat space, distances scale like $e^{(\varphi_h/2)} = f^{1/2}$, while energies scale like $f^{-(1/2)}$, and energy densities like $f^{-2} \sim s^{-2}$ near the nondegenerate RN event horizon.

Requiring the coefficients of the leading s^{-2} and subleading $\ln^2 s$ and $\ln s$ possible divergences in Θ to be zero gives three conditions on the integration constants of the auxiliary fields, just as in the Schwarzschild case, (3.22). A possible s^{-1} divergence in Θ turns out to be linearly dependent on the first three, and is canceled automatically when these three conditions are imposed. Again as in the Schwarzschild case with $L = 0$ there remains a possible s^{-1} divergence in p and ρ unless $C = 0$. Setting $C = 0$ also removes a possible leading s^{-2} behavior in T^{uu}. A fifth and final condition comes from the necessity of canceling the $\ln s$ divergence in the freely falling frame, T^{uu} of (3.17). A possible $\ln^2 s$ term in T^{uu} drops out after we have satisfied the first three conditions for finiteness of $\Theta(r_+)$, and gives no further condition. Hence we end up finally with five algebraic relations on the seven constants of integration for a completely regular static stress tensor in the general RN case, viz.,

$$\ell_0(b'\ell_0 + 2b\lambda_0) = 0 \qquad (s^{-2} \text{ in } \Theta) \tag{3.33a}$$

$$\ell_1(b'\ell_1 + 2b\lambda_1) = 0 \qquad (\ln^2 s \text{ in } \Theta) \tag{3.33b}$$

$$(b'\ell_1 + b\lambda_1)[3 - \epsilon(a_1 + \ell_1) + 2(3\epsilon - 1)\ell_0] + b\ell_1[3\epsilon - \epsilon(c_1 + \lambda_1) + 2(3\epsilon - 1)\lambda_0] = 0 \qquad (\ln s \text{ in } \Theta) \tag{3.33c}$$

$$b(6\epsilon\ell_0 + 6\lambda_0 + 4\epsilon\ell_1\lambda_0 + 4\epsilon\ell_0\lambda_1 - \epsilon\lambda_1) + b'(6\ell_0 - \epsilon\ell_1 + 4\epsilon\ell_0\ell_1) = 0 \qquad (C = 0) \tag{3.33d}$$

$$b[18\epsilon(1-\epsilon)\ell_0 + 9\epsilon(\epsilon - 1)\ell_1 + 3(1 - 4\epsilon + 3\epsilon^2)\lambda_1 - (3 - 12\epsilon + 20\epsilon^2)(\ell_1\lambda_0 + \ell_0\lambda_1)] + b'[3(1 - 4\epsilon + 3\epsilon^2)\ell_1 - (3 - 12\epsilon + 20\epsilon^2)\ell_1\ell_0] = 0 \qquad (\ln s \text{ in } T^{uu}) \tag{3.33e}$$

where ϵ is given by (3.29). The solutions of these conditions will be discussed in the next section.

C. ERN spacetime

When $Q = M$, $r_+ = r_- = M$, the horizon becomes degenerate, and the RN metric function $f(r)$ goes to zero quadratically as $r \to r_\pm$, $(s \to 0)$. This quite different behavior of the spacetime near the horizon is reflected also in the behavior of the conformal differential operator Δ_4 and its solutions. Referring back to (3.28) we observe that when $\epsilon = 0$ the first two terms vanish identically, and the coefficient of the $s^{\gamma+2}$ term becomes the leading one, with coefficient $\gamma(\gamma + 3)$. Thus the more singular behavior $\gamma = -3$ is allowed by (3.28), and the structure of the divergent terms in the solutions to the auxiliary field equations and stress tensor (2.10) becomes quite different in the ERN

limit. For this reason the number of conditions necessary for the stress tensor (2.10) to remain finite increases, and the regularity conditions become more stringent.

Because of the s^{-3} leading behavior allowed by (3.25) for φ' and ψ', the power series expansions of the auxiliary fields in the ERN case are of the form,

$$\varphi = \sum_{n=-2} a_n s^n + \sum_{n=0} \ell_n s^n \ln s \tag{3.34a}$$

$$\psi = \sum_{n=-2} c_n s^n + \sum_{n=0} \lambda_n s^n \ln s \tag{3.34b}$$

instead of (3.30) and (3.32). Substitution of these series into the differential equations for φ and ψ shows that $(a_{-2}, a_{-1}, a_0, a_1)$ and $(c_{-2}, c_{-1}, c_0, c_1)$ are eight free integration constants, with c_0 again playing no role. As in the previous $Q < M$ case, although φ must be strictly static for a time-independent stress tensor, we could allow for nonzero flux by allowing for a nonvanishing linear time dependence η' in ψ. Restricting to static states with $L = 0$ requires $\eta' = 0$. The logarithmic terms and all the other coefficients are related to the set of seven remaining coefficients $(a_{-2}, a_{-1}, a_0, a_1; c_{-2}, c_{-1}, c_1)$ by recursion relations. Hence we have again the same number of free parameters as in the previous two cases with $Q < M$. However, since the leading behaviors of φ and ψ are now s^{-2} as $s \to 0$, the stress tensor (2.10) generally yields a Θ with s^{-4}, s^{-3}, s^{-2}, s^{-1}, $s^{-1}\ln s$, $\ln^2 s$ and $\ln s$ divergent terms. Hence the requirements of a completely regular static stress tensor are more restrictive in the ERN case compared to the previous two cases. The leading s^{-4} behavior again can be understood from the conformal weight of the stress tensor under conformal transformations near the horizon, since $f^{-2} \sim s^{-4}$ in the ERN case.

It turns out that the conditions for removing the leading and all subleading divergent behaviors in the auxiliary field $T^a{}_b$ are not independent, and only *four* independent conditions on the integration constants are sufficient. We can choose these four to be:

$$a_{-2}(2bc_{-2} + b'a_{-2}) = 0 \qquad (s^{-4} \text{ in } \Theta) \tag{3.35a}$$

$$b(-10a_{-2}c_{-2} + a_{-2}c_{-1} + a_{-1}c_{-2}) + b'(-5a_{-2}^2 + a_{-2}a_{-1}) = 0 \qquad (s^{-3} \text{ in } \Theta) \tag{3.35b}$$

$$b[3c_{-1} - 2a_{-2}(9 + 3c_1 + 4c_{-1} - 64c_{-2}) - c_{-2}(36 + 6a_1 + 8a_{-1})] + b'[3a_{-1} - a_{-2}(36 + 8a_{-1} + 6a_1 - 64a_{-2})] = 0 \qquad (s^{-1} \text{ in } \Theta) \tag{3.35c}$$

$$a_{-1}(2bc_{-1} + b'a_{-1}) = 0 \qquad (\ln s \text{ in } \Theta,\ C = 0) \tag{3.35d}$$

In particular the last of these conditions removes all logarithmically divergent terms from Θ, and automatically sets $C = 0$ at the same time.

The requirement (3.17) that T^{uu} be finite with $L = 0$ gives two additional conditions, viz.,

$$b(-15 + 15a_{-1} - 96a_{-2} + 13c_{-1} - 104c_{-2}) + b'(13a_{-1} - 104a_{-2}) = 0 \qquad (s^{-1} \text{ in } T^{uu}) \tag{3.36a}$$

$$b[a_{-2}(486 - 32c_{-1}) + a_{-1}(3c_1 - 72 + 8c_{-1} - 32c_{-2}) + 27 + 3c_{-1}(a_1 - 10) + 288c_{-2}] + b'[4a_{-1}^2 + 288a_{-2} - a_{-1}(30 - 3a_1 + 32a_{-2})] = 0 \qquad (\ln s \text{ in } T^{uu}). \tag{3.36b}$$

Hence we have six apparently independent conditions on the seven active integration constants of the auxiliary fields in order to obtain a fully finite stress tensor from the trace anomaly in the ERN case. We shall find that in fact all six conditions can be satisfied with the choice of only five independent constants. Thus, despite the discontinuously singular behavior of the geometry as $Q \to M$, the more singular behavior of the auxiliary fields in this limit, and the greater restrictiveness of the finiteness conditions (3.35) and (3.36) compared to (3.33), it is still possible for the approximation algorithm for the stress tensor based on the anomalous effective action (2.4) to yield a fully finite stress tensor on the ERN horizon. This is qualitatively different from all previous approximation schemes.

IV. COMPARISON WITH PREVIOUS APPROXIMATIONS AND NUMERICAL RESULTS

Since the method of computing the stress tensor from the anomaly action with the use of auxiliary fields is relatively new, it is interesting to compare it to previous approximation methods, as well as direct numerical evaluations of the renormalized $\langle T^a{}_b \rangle$ whenever the latter are available. An analytic approximation to $\langle T^a{}_b \rangle$ for thermal states of conformal fields in nonconformally flat static spacetimes was derived by Page [19], Brown and Ottewill [20], and Brown, Ottewill and Page [21]. This approximation is based upon the semiclassical approximation to the proper time heat kernel [37], and the properties of the exact one-loop effective action under the conformal transformation,

$$g_{ab} = e^{2\omega}\tilde{g}_{ab}. \tag{4.1}$$

For a classical conformal field the dependence of the exact effective action upon ω is determined completely by the trace anomaly in the form,

$$S_{\rm exact}[g] = S_{\rm exact}[\tilde{g}] + bA[\omega; g] + b'B[\omega; g], \tag{4.2}$$

where $A[\omega; g]$ and $B[\omega; g]$ are known (nonlinear) functionals of ω and g_{ab} which are given in [20,21]. If the conformal transformation $e^{2\omega}$ is chosen to be $f(r)$, for the static, spherically symmetric line element (3.1), then the conformally transformed metric $\tilde{g}_{ab}$ becomes the ultrastatic, optical metric, for which $\tilde{g}_{tt} = -1$. If, in addition, the original physical metric g_{ab} is that of a static Einstein space (i.e. one for which $R_{ab} = \Lambda g_{ab}$), then the invariants appearing in the anomaly,

$$\frac{2}{\sqrt{-g}} g_{ab} \frac{\delta A}{\delta g_{ab}} = \left[F + \frac{2}{3}\Box R \right]_{g=\tilde{g}} = 0, \tag{4.3a}$$

$$\frac{2}{\sqrt{-g}} g_{ab} \frac{\delta B}{\delta g_{ab}} = E|_{g=\tilde{g}} = 0, \tag{4.3b}$$

vanish for the conformally related ultrastatic metric $\tilde{g}_{ab}$, and the A and B terms in (4.2) reproduce the correct trace for the physical metric g_{ab}. Thus, with this choice of ω it is consistent simply to neglect $S_{\rm exact}[\tilde{g}]$ in (4.2), and approximate the full $\langle T^a{}_b \rangle$ by the terms coming from $A[\omega; g]$ and $B[\omega; g]$ which are known analytically.

Because it is also based on the form of the trace anomaly, the Page-Brown-Ottewill (PBO) approximation is related to the approximation scheme based on the auxiliary field effective action of the anomaly (2.10). Indeed, by making use of the conformal transformation property,

$$\sqrt{-\tilde{g}}\tilde{R}^2 = \sqrt{-g}[R + 6(\Box\omega - g^{ab}\partial_a\omega\partial_b\omega)]^2 \tag{4.4}$$

it is not difficult to show that

$$bA[\omega; g] + b'B[\omega; g] = -\Gamma_{\rm WZ}[g; -\omega] + \frac{(b+b')}{18}\int d^4x(\sqrt{-\tilde{g}}\tilde{R}^2 - \sqrt{-g}R^2), \tag{4.5}$$

where

$$\Gamma_{\rm WZ}[g; -\omega] = S_{\rm anom}[g] - S_{\rm anom}[\tilde{g} = e^{-2\omega}g] \tag{4.6}$$

is the Wess-Zumino effective action for the anomaly obtained in [2,7]. Since $\Gamma_{\rm WZ}$ is a quadratic functional of ω, the relation (4.5) shows that the complicated cubic and quartic terms of the PBO effective action are simply the result of adding an R^2 term to $\Gamma_{\rm WZ}$, with a corresponding $b''\Box R$ term in the trace anomaly (2.1). The PBO effective action $W_{\rm PBO}[g]$ is related then to the anomaly effective action $S_{\rm anom}$ of (2.4) by

$$W_{\rm PBO}[g] = W_{\rm PBO}[\tilde{g}] + \frac{(b+b')}{18}\int d^4x\sqrt{-\tilde{g}}\tilde{R}^2 - S_{\rm anom}[\tilde{g}] + S_{\rm anom}[g] - \frac{(b+b')}{18}\int d^4x\sqrt{-g}R^2. \tag{4.7}$$

This is consistent with the decomposition (2.11), in which all terms that depend only upon $\tilde{g}_{ab}$ are viewed as invariant under the local Weyl transformation (4.1). Because of the different linear combination of invariants in (4.3) from the b and b' terms in (2.1), the PBO conformal transformation is given by an ω which satisfies

$$4\Delta_4\omega = E - F - \frac{2}{3}\Box R, \tag{4.8}$$

and therefore corresponds to a particular linear combination (i.e. $\varphi - \psi$) of the auxiliary fields, with a particular choice of homogeneous solution.

Although the PBO approximation is related to the one based on $S_{\rm anom}$, there are two important differences. First, in the PBO approximation the conformal transformation $e^{2\omega}$ to the ultrastatic metric is *fixed* up to a linearly time dependent term in ω, by the requirement that the invariants in (4.3) vanish in the conformally transformed spacetime. Hence it contains only one free parameter, namely, the coefficient of linear time dependence free in ω, which is far fewer than the seven integration constants corresponding to the freedom to adjust the state-dependent and Weyl invariant part of the effective action in the auxiliary field method. Second, and even more importantly, the invariants in (4.3) vanish in the conformally transformed spacetime only in rather special cases, such as if the original metric g_{ab} is that of a static Einstein space. Although Zannias suggested a modification of the PBO ansatz to account for the correct trace in non-Einstein spaces [22], the nonvanishing of the trace in the conformally related space with metric $\tilde{g}_{ab}$ means that the original PBO rationale for ignoring $S_{\rm exact}[\tilde{g}]$ in (4.2) no longer applies. Hence there is no especially good reason why this modification of the PBO approximation should be accurate, or even finite on the event horizon. If one tries instead to find a spacetime where the conditions (4.3) for the vanishing of the trace are satisfied, then one is faced with solving *two* nonlinear conditions for a single ω. Hence one would not expect that a simultaneous solution of both conditions (4.3) exists at all for $\tilde{g}_{ab} = e^{-2\omega}g_{ab}$, for general g_{ab}.

In the auxiliary field method, these difficulties are removed completely. The *two* independent φ and ψ fields satisfy the *linear* equations (2.8), for which solutions are guaranteed to exist and can be found in *any* spacetime. Although $e^{-\varphi}$ may be regarded as a conformal transformation to a spacetime where $E - \frac{2}{3}\Box R$ vanishes, there is no such conformal interpretation for the second auxiliary field ψ, and none is required. The auxiliary scalars and their stress tensor simply encode the same information about the

full nonlocal trace anomaly in a local, generally covariant form, and there is no conformally related $\tilde{g}_{ab}$ enjoying a privileged status over any other. This is clear also from the freedom to add arbitrary homogeneous solutions to (2.8), corresponding to different choices for the conformal image $\tilde{g}_{ab}$, and different Weyl invariant parts of the effective action. This additional freedom in the linear system (2.8) makes an exploration of a much wider class of states possible with the auxiliary field algorithm, with different state-dependent traceless contributions to the stress tensor, all consistent with the correct trace anomaly. It is of course the much larger parameter space available in the auxiliary field method which makes it possible to find regular stress tensors on both the RN and ERN event horizons.

The PBO approximation was later rederived in a different way by Frolov and Zel'nikov (FZ), by carrying out an analysis of possible terms in the effective action in general spacetimes with a static Killing vector field, $K = \frac{\partial}{\partial t}$ [23]. Although rather different in methodology, the FZ approach also applies only to static spacetimes and makes use of the conformal transformation properties (conformal weights) of the various terms in the stress tensor. As a result, although the FZ approach is not limited *a priori* to Einstein spaces, the effective action they obtain is in fact equivalent to that of PBO (as modified by Zannias), up to local $F = C_{abcd}C^{abcd}$ and the $(\sqrt{-\tilde{g}}\tilde{R}^2 - \sqrt{-g}R^2)$ terms appearing in (4.5), which are allowed to have arbitrary coefficients unrelated to the anomaly. The first of these particular Weyl invariant terms is mildly behaved on the event horizon whereas divergences in the second amounts to readjustment of the coefficient of linear time dependence in ω of the PBO approximation. Therefore neither of the terms can be used to cancel any divergences present in the PBO approach. Hence when applied to the non-Ricci flat Reissner-Nordström geometry, the FZ approximation suffers from the same limitation as that of PBO, namely, both predict a logarithmically divergent stress tensor on the RN event horizon, and both a linear and a logarithmic divergence on the ERN horizon.

From our present vantage point, the interesting feature of the FZ approach is the central role of the static Killing field K_a. The FZ approach underlines the fact that $\langle T^a{}_b \rangle$ generally depends upon *global* invariant functions of K_a as in (3.18), in addition to strictly local invariants such as F, E and R^2. Since global invariants such as (3.18) may diverge on the event horizon, and $\langle T^a{}_b \rangle$ generally is a function of these invariants, explicitly so in the FZ approach, it is clear that requiring such divergences to be absent in $\langle T^a{}_b \rangle$ is a dynamical restriction on the quantum state, not at all required by general coordinate invariance.

It was shown by Howard and Candelas [12,13] for the conformally invariant spin 0 field in Schwarzschild spacetime that the WKB approximation can be used to write the stress-energy tensor in terms of the PBO approximation for each field plus a term containing mode sums that must be computed numerically. A similar result was obtained for the massless spin 1 field by Jensen and Ottewill [14]. Anderson, Hiscock, and Samuel [16] (AHS) showed that if the WKB approximation in the high frequency limit is used for the radial modes of the Euclidean Green's function for a massless scalar field with arbitrary curvature coupling in a general static spherically symmetric spacetime, then a conserved stress-energy tensor results which in the case of conformal coupling has a trace equal to the trace anomaly. For the case of conformal coupling this stress-energy tensor is equivalent to that derived by FZ if the three arbitrary constants in their derivation are set to zero. A similar approximation was derived by Groves, Anderson, and Carlson [24] (GAC) for the massless spin 1/2 field. In this case the approximation is equivalent to that of FZ if their arbitrary constants have the values $q_1^{(0)} = q_2^{(0)} = 0$ and $q_1^{(2)} = 1/144$.

Huang's evaluation of the stress tensor for a conformally invariant scalar field in the PBO/FZ approximation [38], apparently correcting an error in [22] for the ERN case, shows this same logarithmic divergence of (3.16) for $Q < M$, which becomes a combination of a linear and logarithmic divergence in the ERN case.[3] Not surprisingly the same divergences were found for the AHS and GAC approximations [16,18]. Thus all preexisting analytic approximations to $\langle T^a{}_b \rangle$ diverge at least logarithmically in the general RN spacetime, and linearly in the ERN case.

A. Schwarzschild spacetime

The six conditions (3.24) guaranteeing finiteness of the stress tensor on the Schwarzschild horizon can be satisfied in several different ways. The simplest possibility is [2]

$$(2b + b')c_H^2 + \eta(2b\eta' + b'\eta) = 0, \tag{4.9a}$$

$$bd_H = -(b + b')c_H, \tag{4.9b}$$

$$q = q' = 2, \tag{4.9c}$$

$$b'q\eta + b(q\eta' + q'\eta) = 0, \tag{4.9d}$$

which requires fixing only five parameters to satisfy all six conditions (3.22), (3.23), and (3.24), with $L = 0$. This is because the choice $q = q' = 2$ eliminates all the logarithmic terms in the auxiliary fields at the horizon (hence all $\ln^2 s$ and $\ln s$ terms in $T^a{}_b$) and simplifies the remaining conditions considerably. The three parameter subset of the original eight parameter family of spherically symmetric auxiliary fields is certainly not generic, but still leaves considerable freedom to fit the finite values of the stress tensor at $r = 2M$ and/or $r = \infty$ in the regular Hartle-Hawking-Israel state [26]. This is the choice which was

[3]The third footnote of Ref. [38] gives an incorrect criterion for finiteness of the stress tensor on the horizon in freely falling coordinates, replacing one factor of $f(r)$ in (3.16) by $f^{1/2}(r)$. For this reason the conclusions drawn in [38] are not warranted by the evaluation of $T^a{}_b$ given.

studied in some detail in [2]. The other possible ways of solving the finiteness conditions are listed in the appendix. Approximate stress tensors for the Unruh state [39] may be obtained as well by replacing the $L = 0$ condition with the properly normalized luminosity (3.23), choosing η and η' appropriately in the Schwarzschild case, and just $\eta = 0$, $\eta' \neq 0$ in the $Q > 0$ cases.

The PBO/FZ approximation works quite well for the conformally invariant scalar field in the Hartle-Hawking-Israel state in Schwarzschild spacetime, although the method based on the auxiliary field effective action of the anomaly accounts for the behavior of the stress tensor more accurately in states with divergent stress tensors on the horizon, such as the Boulware state in Schwarzschild spacetime. For the spin 1/2 and spin 1 fields the PBO/FZ approximation works less well [14,18].

Balbinot, Fabbri, and Shapiro [6] (BFS) also used two auxiliary fields to study the properties of the approximate stress-energy tensor based on the anomaly in Schwarzschild spacetime. Thus their approach comes the closest of any previous one to the present work. Their two auxiliary fields (ϕ, ψ) are related to ours by the linear combinations, $-\sqrt{-b'}\varphi + b\psi/\sqrt{-b'}$ and $b/\sqrt{-b'}\psi$ respectively. However, BFS applied conditions to each of these two linear combinations of auxiliary fields separately, rather than searching for the more general solutions of the finiteness conditions (3.22) and (3.24) on the Schwarzschild horizon. This amounts to fixing the traceless terms in the stress tensor with fewer free parameters than in the present approach. For this reason they were not able to reproduce some features of the exact stress tensor in the Hartle-Hawking-Israel state, such as the correct value of $\langle T^a{}_b \rangle$ on the Schwarzschild event horizon or at infinity. In the case of the present formulation, after all four finiteness conditions (3.22) and (3.24) are satisfied, there remains a three parameter family of finite stress tensors. Hence both the values on the horizon and at infinity can be adjusted to their correct values by a suitable choice of the integration constants.

As pointed out by BFS, the subdominant terms in the stress tensor at infinity are also not reproduced by the stress tensor of the anomaly, and this undesirable feature persists in our approach. Clearly this is because the anomaly action is not equal to the full quantum effective action, differing from it by Weyl invariant terms, as in (2.11). In regular states these give rise to additional traceless terms in the stress tensor, which would be expected to be of the same order as those in (2.10). One possibility for improvement is to add the Weyl invariant term,

$$S_{\rm inv}[g;\chi] = k\int \sqrt{-g}d^4x\left(-\frac{1}{2}\chi\Delta_4\chi + F\chi\right) \qquad (4.10)$$

to the total effective action. If we added this term to $S_{\rm anom}$, we would have a third auxiliary field, denoted here by χ, with four more integration constants to serve as free parameters in the stress tensors of the RN and ERN spacetimes. Although this may allow for more accurate fitting of the numerical results for $\langle T^a{}_b \rangle$ in the RN and ERN cases, we do not pursue this possibility here, and thus set $k = 0$ in our approximation.

B. General RN spacetime

When the PBO/FZ, AHS, and GAC approximations are applied to the $Q > 0$ RN spacetimes, they predict that $\langle T^a{}_b \rangle$ diverges logarithmically on the RN event horizon for $Q < M$ and linearly and logarithmically for $|Q| = M$. The reason for the divergence of the PBO/FZ approximation is that the semiclassical approximation for the heat kernel of [37] fails as $r \to r_+$. Thus for massless fields all previous approximations are both highly specialized to certain specific classes of spacetimes, and also lead inevitably to divergences on the event horizons of non-Einstein static spaces, such as the Reissner-Nordström metric. The present approach based on the auxiliary field form of the effective action of the anomaly does not rely on a WKB approximation, and hence does not suffer from these limitations.

The direct method of evaluating $\langle T^a{}_b \rangle$ from the wave equations of the underlying conformal field theory has been carried out for both scalar and spinor fields in RN spacetimes with $0 < Q \leq M$ [16–18]. These direct evaluations show no evidence of a divergence as the horizon is approached. In the following plots we show the results of these exact numerical evaluations of $\langle T^a{}_b \rangle$ and the approximation of the present paper for RN spacetimes with $|Q|/M = 0.99$ and 0.8.

The approximation based on the auxiliary field effective action is qualitatively better than that of PBO/FZ in all $Q > 0$ RN regular states in that it allows for a finite stress tensor on the event horizon, showing roughly comparable features to those already encountered in the Schwarzschild case in [2]. The main difference from the $Q = 0$ Schwarzschild case compared to the $0 < Q \leq M$ RN cases is the loss of the integration constant η in the latter cases, since such a linear time dependence in φ leads to time dependence in the stress tensor (2.10) in the non-Ricci flat RN geometries. Since a constant a_0 in φ appears in the stress tensor when $R_{ab} \neq 0$, there are exactly seven free integration constants for regular static stress tensors in all RN cases with vanishing flux, $L = 0$.

In the present approximation algorithm there are various possible ways of solving the five finiteness conditions (3.33). For example, the first condition (3.33a) is solved by either

$$\ell_0 = 0 \qquad (4.11a)$$

or

$$\ell_0 = -\frac{2b}{b'}\lambda_0. \tag{4.11b}$$

Likewise, independently of this choice, the second condition (3.33b) is solved by either

$$\ell_1 = 0 \tag{4.12a}$$

or

$$\ell_1 = -\frac{2b}{b'}\lambda_1. \tag{4.12b}$$

When the remaining three conditions are considered, it becomes apparent that either λ_0 and λ_1 are both zero, or they are both nonzero. The first option, viz.

$$\ell_0 = \lambda_0 = \ell_1 = \lambda_1 = 0, \qquad \text{(minimal)} \tag{4.13}$$

we term "minimal" because it requires only four integration constants be set to zero in order to satisfy all five finiteness conditions (3.33). Since this leaves the most integration constants still free to adjust, we consider this minimal solution of the finiteness conditions in order to compare the auxiliary field algorithm to the direct evaluation of $\langle T^a{}_b \rangle$. The other possibilities for solving the conditions (3.33) are catalogued in the appendix.

The remaining three nontrivial integration constants are a_0, a_1, and c_1. If the stress tensor is finite on the horizon then $T^r{}_r = T^t{}_t$ there. So once the value of $T^t{}_t$ is known, the value of $T^\theta{}_\theta$ on the horizon is fixed by the value of the trace anomaly there. One can compute the components of the approximate stress tensor on the horizon from the general expressions (2.6), (2.7), and (2.10) by using the metric (3.1) and metric functions (3.9) along with the expansions (3.30) and (3.32). One finds that the value of $T^t{}_t$ on the horizon depends on a_1 and c_1 and the value of T^{uu} on the horizon depends on a_0, a_1 and c_1.

A numerical code was developed which for a given value of a_1 chose the values of a_0 and c_1, so that $T^t{}_t$ and T^{uu} matched the values previously obtained [16,18] from exact numerical computations of the stress tensor on the horizon for the spin 0 and spin 1/2 fields. This code then solves the equations for the auxiliary fields ϕ and ψ and computes the analytic approximation (2.10) for the stress tensor for various values of the radial coordinate r. Different values of a_1 lead to different behaviors at large values of r. The goal is to find the value of a_1 which gives the same energy density at large r as the field has if it is in the Hartle-Hawking-Israel state. This is the best that one can do to find an approximation for the stress energy which is finite on the horizon and has the correct energy density at large values of r. Of course one could take the opposite approach and fix the behavior at large r more accurately, but then the stress energy would not be regular on the event horizon.

We have examined the cases $|Q| = 0.99M$ and $0.8M$ for the conformal scalar and spin 1/2 fields, and found in each case two values of a_1 that satisfy the above criterion. In each case one of these values gives results in closer agreement with the numerically computed stress-energy tensor, so we have chosen that one. The results for the spin 0 field are shown in Fig. 1 and the corresponding comparisons for the massless spin 1/2 Dirac field are shown in Figs. 2. It can be seen that the approximation based on the pure anomaly action is quite accurate close to the horizon, for

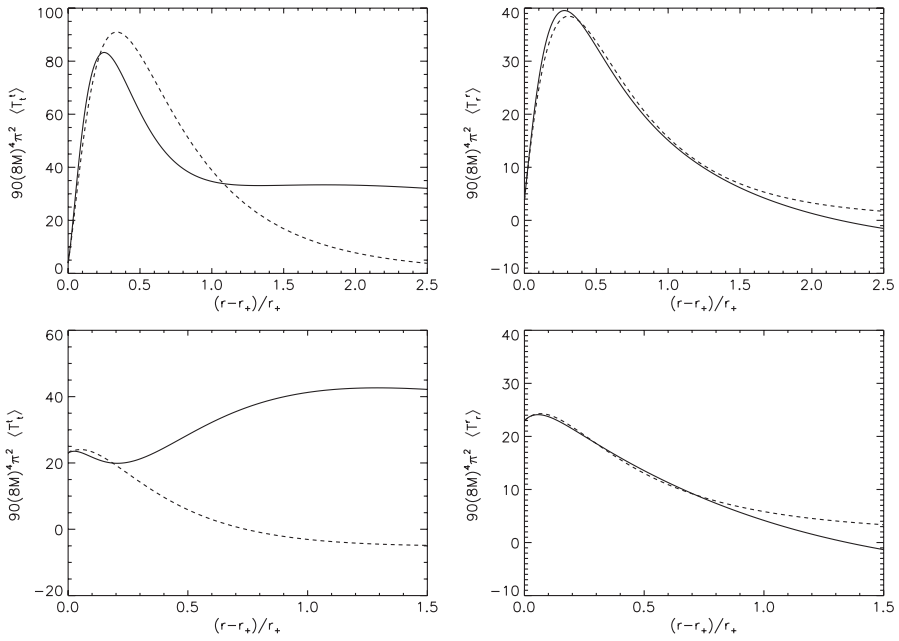

FIG. 1. The top two plots show the expectation values $\langle T^t{}_t \rangle$ and $\langle T^r{}_r \rangle$ for the conformally invariant scalar field in the Reissner-Nordström geometry with $|Q| = 0.99M$. The bottom two plots show these components for the case $|Q| = 0.8M$. In all plots the solid line corresponds to the auxiliary field stress tensor and the dashed line to the numerically computed exact one.

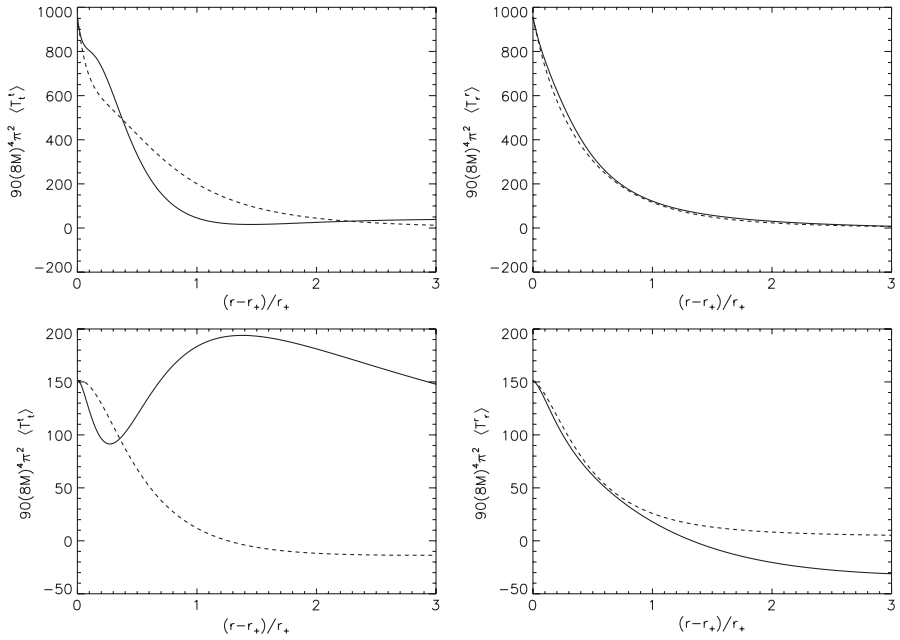

FIG. 2. The top two plots show the expectation values $\langle T^t{}_t \rangle$ and $\langle T^r{}_r \rangle$ for the massless spin 1/2 field in the Reissner-Nordström geometry with $|Q| = 0.99M$. The bottom two plots show these components for the case $|Q| = 0.8M$. In all plots the solid line corresponds to the auxiliary field stress tensor and the dashed line to the numerically computed exact one.

$r \lesssim 1.2r_+$, becoming less accurate at intermediate values of $r \sim 2r_+$. At larger values of r, the components of the approximate stress tensor do not approach their asymptotic values nearly as quickly as the numerically computed values. All the plots should be viewed as qualitative, for illustration purposes only. We have not performed a systematic quantitative study of the stress tensor based on the anomaly as a function of r for a fixed value of $|Q|/M$, studied the detailed dependence on spin, nor attempted to quantify the trends as a function of $|Q|/M$.

C. ERN spacetime

Finally in the ERN case of $|Q| = M$ the tensor (2.10) is still able to return a completely finite approximation to $\langle T^a{}_b \rangle$ at $s = 0$, provided that all six conditions in Eqs. (3.35) and (3.36) are fulfilled. There are three algebraically distinct ways of solving these equations, which are described in detail in the appendix. The simplest possibility is to require that $E^a{}_b$ and $F^a{}_b$ be separately finite. In that case we find that the five values of the coefficients in terms of rational fractions,

$$a_{-2} = 0, \tag{4.14a}$$

$$a_{-1} = 0, \tag{4.14b}$$

$$a_1 = -\frac{9}{5}, \tag{4.14c}$$

$$c_{-2} = \frac{75}{26}, \tag{4.14d}$$

$$c_{-1} = \frac{315}{13}, \tag{4.14e}$$

satisfy all six finiteness conditions (3.35) and (3.36).

Despite having two remaining integration constants free, this and the other two solutions of the finiteness conditions (3.35) and (3.36) in the ERN case are rather restrictive. It turns out that for all three of the distinct solutions catalogued in the appendix, the diagonal components $T^t{}_t$, $T^r{}_r$, and $T^\theta{}_\theta$ on the horizon are all fixed (for fixed values of b and b') to definite finite values. For example, in the case of the solution (4.14) these components have the specific finite values,

$$T^t{}_t = T^r{}_r = \frac{178}{13M^4} b - \frac{2}{M^4} b' \tag{4.15a}$$

$$T^\theta{}_\theta = -\frac{178}{13M^4} b - \frac{2}{M^4} b'. \tag{4.15b}$$

On the other hand the exact values of these components on the ERN horizon in regular states are [40]

$$T^t{}_t = T^r{}_r = T^\theta{}_\theta = -\frac{2}{M^4} b'. \tag{4.16}$$

The mismatch of the first term proportional to b in both members of (4.15) accounts for the large discrepancy of the approximation from the numerical data apparent in Figs. 3 and 4. This indicates that we are certainly lacking a term from S_{inv} in the exact effective action in our minimal approximation based on the anomaly, which is needed to give the correct finite coefficient of the stress tensor on the ERN horizon. The term (4.10) is an example of a possible term which we have not considered. That the missing term of S_{inv} is apparently related to the b coefficient of the anomaly, as suggested by the comparison of (4.15) with

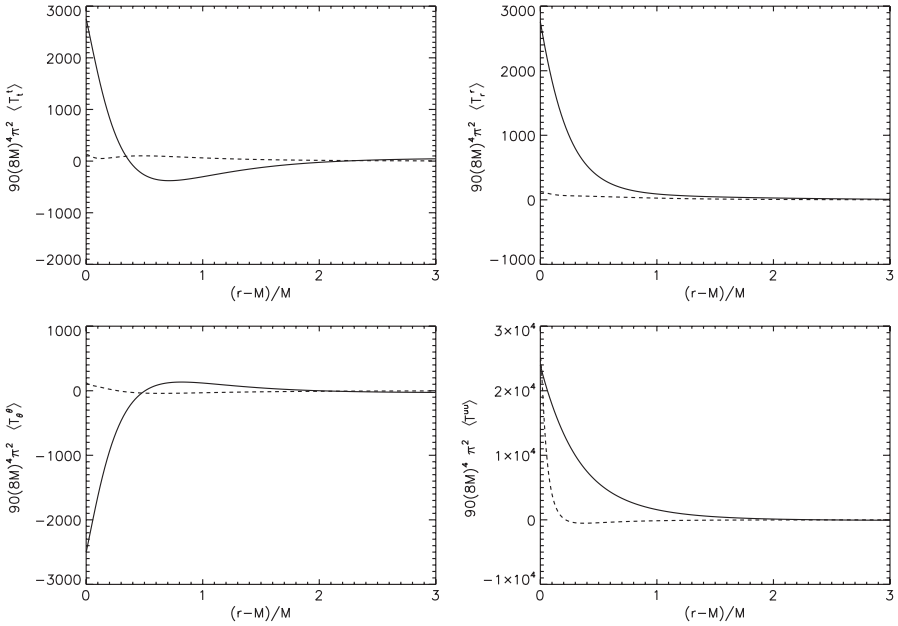

FIG. 3. The top two plots show the expectation values $\langle T^t{}_t \rangle$ and $\langle T^r{}_r \rangle$ for the conformally invariant scalar field in the Reissner-Nordström geometry with $|Q| = M$. The bottom two plots show the $\langle T^\theta{}_\theta \rangle$ and $\langle T^{uu} \rangle$ components. In all plots the solid line corresponds to the auxiliary field stress tensor and the dashed line to the numerically computed exact one.

(4.16), with the latter determined by a conformal mapping from flat space, suggests that the additional term might be determined by an extension of our effective action S_{anom} based on the conformal anomaly.

On the other hand, the value of the T^{uu} component on the horizon is not fully determined by (4.14) and instead depends on a_0 and c_1 via

$$T^{uu} = \frac{1}{585M^4}(306\,856 - 9360a_0 + 12\,168c_1)b - \frac{4484}{75M^4}b' \tag{4.17}$$

which are the still remaining two free parameters. Hence it is possible to adjust the value of the T^{uu} component on the

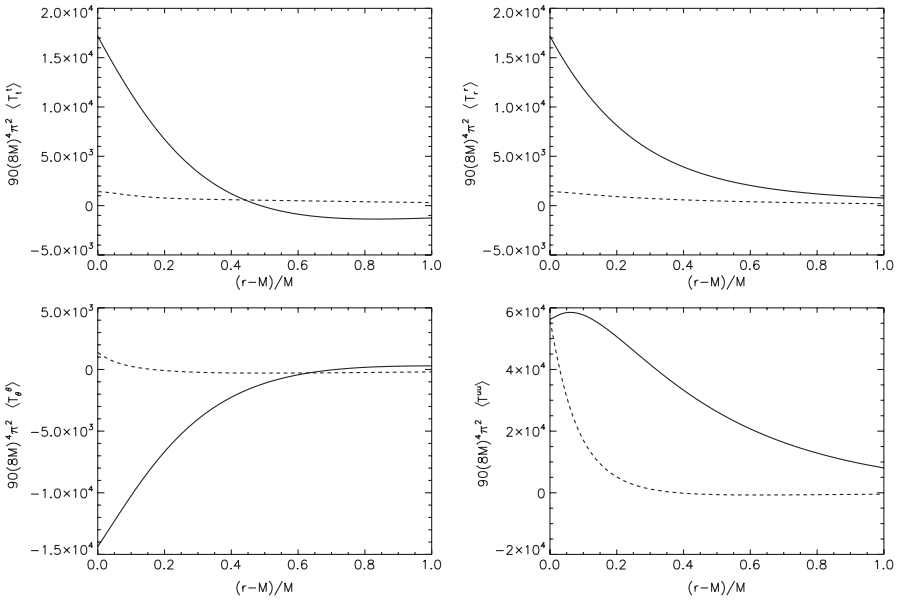

FIG. 4. The top two plots show the expectation values $\langle T^t{}_t \rangle$ and $\langle T^r{}_r \rangle$ for the massless spin $1/2$ field in the Reissner-Nordström geometry with $|Q| = M$. The bottom two plots show the $\langle T^\theta{}_\theta \rangle$ and $\langle T^{uu} \rangle$ components. In all plots the solid line corresponds to the auxiliary field stress tensor and the dashed line to the numerically computed exact one.

horizon to any finite value and still have one free parameter left unspecified. This last parameter can be fixed by requiring that the behavior of all components of the stress tensor vanish as $r \to \infty$, consistent with a zero temperature state in an asymptotically flat spacetime.

A numerical code was developed which for a given value of c_1 chooses the value of a_0 so that T^{uu} matched the values previously obtained [16,18] from exact numerical computations of the stress tensor on the horizon for the spin 0 and spin $\frac{1}{2}$ fields. As in the RN case described above, this code then solves the equations for the auxiliary fields ϕ and ψ and computes the analytic approximation (2.10) for the stress tensor for various values of the radial coordinate r.

Our results for the conformally invariant spin 0 field are shown in Fig. 3 and those for the massless spin $\frac{1}{2}$ field are shown in Fig. 4. Since (4.15) and (4.16) shows that the $T^t{}_t$, $T^r{}_r$, and $T^\theta{}_\theta$ components are not accurate at the horizon, the plots show that this inaccuracy persists at intermediate values of r. Since it is possible to choose parameters to fit the correct value of T^{uu} on the horizon, the approximation works better for that component. Because of the more severe horizon divergences possible in the ERN case, the finiteness conditions (3.35) and (3.36) eliminating them are restrictive enough to lead to stress tensors which are not very good approximations to the numerical results obtained by the direct method. One may consider relaxing the finiteness condition on the logarithmic terms in T^{uu}, still requiring no power law divergences in s as $s \to 0$. In this case it is possible to obtain the correct values of $T^t{}_t$, $T^r{}_r$, and $T^\theta{}_\theta$ very near the horizon. However we have found that the approximation remains poor at intermediate and large values of r and of course it also gives a divergence in T^{uu} on the horizon, for which the numerics shows no evidence.

A more promising approach is to add additional terms in $S_{\rm inv}$, as for example (4.10), in an attempt to find a better approximation that is finite on the horizon. Although a modification such as this is very likely to introduce enough new parameters to allow for significant improvement of the comparisons with the numerical results in Figs. 3 and 4, we did not pursue this possibility, preferring to put the minimal two field anomaly action and stress tensor to its most stringent test. We also did not add other possible terms to the effective action, such as the last term in Eq. (4.5), which gives contributions to $\langle T^a{}_b \rangle$ of the same order as the anomaly action in regular states (where all contributions are of order M^{-4} and small for $M \gg M_{Pl}$). While the approximation based solely on the anomaly action (2.4) and (2.5) itself is not uniformly accurate for RN spacetimes in such regular states, it is worth emphasizing that it is the first finite approximation that has been obtained for massless quantized fields in these spacetimes, and therefore captures an element of the exact effective action lacking in previous approaches, even without the improvements possible with the addition of specific Weyl invariant terms.

V. CONCLUSIONS

The effective action associated with the trace anomaly provides a general algorithm for approximating the expectation value of the stress tensor of conformal matter fields in arbitrary curved spacetimes. It successfully classifies the leading and subleading divergent behaviors of the quantum stress tensor components in the vicinity of all spherically symmetric event horizons, and the conformal properties of horizons. These behaviors follow from an analysis of the allowed singular behaviors of the auxiliary fields as the event horizon is approached, which are determined by solutions of the conformally invariant differential operator (2.9).

A numerical solution of the auxiliary field equations is necessary to construct the stress tensor at points arbitrarily distant from the horizon, except for the uncharged Schwarzschild case which admits a completely analytic solution. Because of the possibility of adding homogeneous solutions to the auxiliary field equations of motion (2.8), the generic divergent behaviors may be canceled, and finite approximations to $\langle T^a{}_b \rangle$ obtained on all charged RN event horizons, including also the extremal case of $|Q| = M$.

Although it is possible to constrain the solutions of the auxiliary field equations by the requirement that the stress tensor should be regular on the horizon for all Q, we emphasize that the fine tuning of integration constants that is necessary to achieve this suggests that a regular stress tensor is very much the nongeneric case. This is in accord with the well-known fact that to have a completely regular static stress-energy tensor in a static black hole spacetime it is necessary for the field to be in a Hartle-Hawking-Israel state [26] which is a thermal state at the black hole temperature. What may be less well known is that this is also true for the zero temperature ERN black hole [17] for which the Hartle-Hawking-Israel and Boulware [36] states coincide. Also the fact of having a thermal state at a given temperature (including zero) does not completely specify the state. For example one can put boundary conditions on the mode functions at some particular value of the radial coordinate r as is done when a spherical mirror surrounding the black hole is present [11,41].

The generic diverging behavior of the scalar auxiliary fields and their associated stress tensor near the event horizon has a geometric origin in the behavior of the Killing field of the static geometry becoming null on the horizon, through (3.18). Thus, possible divergences of the stress tensor on the horizon are perfectly consistent with the equivalence principle, notwithstanding the finiteness of local curvature invariants such as R or $R_{abcd}R^{abcd}$ there.

The comparison of the anomaly induced stress tensor, viewed as an approximation to $\langle T^a{}_b \rangle$ in Eq. (2.10), with the direct evaluation of the renormalized expectation value for fields of spin 0 and $\frac{1}{2}$ shows that it yields a fair approxima-

tion for states which are regular on the horizon, becoming less accurate quantitatively both away from the horizon and in the case of the degenerate horizon of the extreme $|Q| = M$ case. Although the finite terms may be more accurately obtained by a better understanding of the Weyl invariant nonlocal terms in the quantum effective action, such as (4.10), the fact that the minimal anomalous terms can give all the possible divergent or regular behaviors of the stress tensor in the vicinities of all RN black hole event horizons allows us to regard it as a candidate geometrical action for meaningful backreaction calculations. Since the effective action and auxiliary field stress tensor of the anomaly can be computed in principle in any spacetime without regard to special symmetries, backreaction calculations are possible with $S_{\rm anom}$ of (2.4) added to the classical Einstein-Hilbert and matter field actions. Moreover, since the resulting system of equations can be treated by classical methods, dynamical backreaction calculations in time dependent and even nonspherically symmetric spacetimes undergoing gravitational collapse, taking into account one-loop vacuum polarization and particle creation effects, would seem to be practically feasible by this approach for the first time.

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APPENDIX: GENERAL SOLUTIONS OF THE STRESS TENSOR FINITENESS CONDITIONS

In this appendix we list the possible ways for solving the horizon finiteness conditions in each of the three cases considered in the text.

1. Schwarzschild spacetime

The solution of the regularity conditions in Schwarzschild spacetime can be classified into three different groups, depending upon the way one solves (3.22b).

The first group is special, because one can satisfy the three conditions (3.22), by specifying only two of the free constants, namely $q = q' = 2$. This is the solution considered in [2]. All of the finiteness conditions can be satisfied by specifying only five integration constants viz.,

$$q = 2 \tag{A1a}$$

$$q' = 2 \tag{A1b}$$

$$2bd_H = (bq\eta' + bq'\eta + b'q\eta) - 2(b + b')c_H \tag{A1c}$$

$$2b\eta' = (bq\eta' + bq'\eta + b'q\eta) - 2(b + b')\eta \tag{A1d}$$

$$c_H = \eta \quad \text{or} \quad c_H = \frac{(bq\eta' + bq'\eta + b'q\eta)}{2b + b'} - \eta. \tag{A1e}$$

The other two groups of solutions use six instead of five constants to satisfy the regularity conditions:

$$q = 2 \tag{A2a}$$

$$c_H = 0 \tag{A2b}$$

$$c_\infty = \frac{20}{9} \tag{A2c}$$

$$d_H = \frac{(bq\eta' + bq'\eta + b'q\eta)}{2b} + \frac{q' - 2}{6} \tag{A2d}$$

$$\eta' = \frac{(bq\eta' + bq'\eta + b'q\eta)}{2b} - \frac{2b' + bq'}{2b}\eta \tag{A2e}$$

$$\eta = 0 \quad \text{or} \quad \eta = \frac{(bq\eta' + bq'\eta + b'q\eta)}{b' + bq'} \tag{A2f}$$

and

$$q = 2 - \frac{2b}{b'}(q' - 2) \tag{A3a}$$

$$c_H = -\frac{2b}{b'}d_H \tag{A3b}$$

$$c_\infty = \frac{20}{9} - \frac{2b}{b'}\left(d_\infty - \frac{20}{9}\right) \tag{A3c}$$

$$d_H = -\frac{b'}{2b(2b + b')}(bq\eta' + bq'\eta + b'q\eta) + \frac{b'(q' - 2)}{6(2b + b')} \tag{A3d}$$

$$\eta' = \frac{b'}{2b}\left(\frac{(bq\eta' + bq'\eta + b'q\eta) - \eta(4b + 2b' - bq')}{2b + b' - bq'}\right) \tag{A3e}$$

$$\eta = 0 \quad \text{or} \quad \eta = \frac{(bq\eta' + bq'\eta + b'q\eta)}{2b + b'}. \tag{A3f}$$

In the last two groups it is assumed that $q' \neq 2$. Note that with all of the above set of constants the regularity con-

ditions for the stress-energy tensor are satisfied for any luminosity L, which is given by

$$L = \frac{\pi}{M^2}(bq\eta' + bq'\eta + b'q\eta). \tag{A4}$$

This combination of parameters appears in several places in the above solutions, so that if L is set equal to zero, as it is in the Hartle-Hawking-Israel state, considerable simplification of the relations above results.

2. General RN spacetime

The simplest possibility for solving all five conditions (3.33) in the $0 < Q < M$ cases is the minimal one (4.13). It requires the fixing of only four integration constants whose values are given in Eq. (4.13). The other two distinct possibilities are:

$$\ell_0 = 0 \tag{A5a}$$

$$\ell_1 = -\frac{2b}{b'}\lambda_1 \tag{A5b}$$

$$\lambda_1 = \frac{6b'\lambda_0}{\epsilon(8b\lambda_0 - b')} \tag{A5c}$$

$$a_1 = \frac{3}{\epsilon} - \frac{2b}{b'\epsilon}[(c_1 - 3)\epsilon + 2\lambda_0(1 - 3\epsilon)] \tag{A5d}$$

$$\lambda_0 = \frac{3(1-\epsilon)(-6b\epsilon + (1-3\epsilon)b')}{2b(3 - 12\epsilon + 20\epsilon^2)} \tag{A5e}$$

and

$$\ell_0 = -\frac{2b}{b'}\lambda_0 \tag{A6a}$$

$$\ell_1 = 0 \tag{A6b}$$

$$\lambda_1 = -\frac{6(2b\epsilon + b')\lambda_0}{\epsilon(8b\lambda_0 + b')} \tag{A6c}$$

$$a_1 = \frac{3}{\epsilon} + \frac{4b}{b'\epsilon}\lambda_0(1 - 3\epsilon) \tag{A6d}$$

$$\lambda_0 = -\frac{3b'(1-\epsilon)[2b\epsilon(1-2\epsilon) + b'(1-3\epsilon)]}{2b[2b\epsilon(8\epsilon^2 + 3) + b'(3 - 12\epsilon + 20\epsilon^2)]}. \tag{A6e}$$

These latter two solutions use five constants and are more restrictive than the first minimal one in terms of four integration constants. In particular the values of the components $T^t{}_t$, $T^r{}_r$ and $T^\theta{}_\theta$ are all fixed on the horizon in the latter two solutions. There remains the freedom to adjust T^{uu} on the horizon, which depends upon the value of a_0.

3. ERN spacetime

In addition to the solution (4.14) to the regularity conditions (3.35) and (3.36) given in the text, which yield finite $E^a{}_b$ and $F^a{}_b$ tensors separately, two other distinct solutions of these conditions are possible, viz.

$$a_{-2} = -\frac{2b}{b'}c_{-2} \tag{A7a}$$

$$a_{-1} = -\frac{2b}{b'}c_{-1} \tag{A7b}$$

$$a_1 = -\frac{60b^2(3 + c_1)c_{-1} + 2bb'(45 + 15c_1 + 81c_{-1} + 13c_1c_{-1}) + 2b'^2(45 + 13c_{-1})}{30bb'c_{-1} + b'^2(15 + 13c_{-1})} \tag{A7c}$$

$$c_{-2} = \frac{30bc_{-1} + b'(15 + 13c_{-1})}{8(24b + 13b')} \tag{A7d}$$

$$c_{-1} = \frac{3510b^2b' + 1329b'^2b + 13b'^3 \pm |b'(24b + 13b')|\sqrt{32400b^2 + 4320bb' + b'^2}}{1620b^3 + 3156b^2b' + 1105b'^2b}. \tag{A7e}$$

The two solutions involve a cancellation of divergences on the horizon between the $E^a{}_b$ and $F^a{}_b$ terms of the anomaly, and consequently have a more complicated dependence on b and b'. They differ only in the sign of the square root in the expression for c_{-1}. The common feature of these two additional solutions and the one studied in more detail in the text is that only five out seven available integration constants are used. Nevertheless, as in the previous RN case, the values of the components $T^t{}_t$, $T^r{}_r$ and $T^\theta{}_\theta$ are fixed on the horizon for all the finite solutions. The values on the horizon for this second set of solutions (A7) are

$$\begin{aligned} T^t{}_t = T^r{}_r = [&13275b^2b'^2 + 7740bb'^3 - 1170b'^4 \\ &+ 31500b^3b'c_{-1} + 27870b^2b'^2c_{-1} + 2808bb'^3c_{-1} \\ &- 1014b'^4c_{-1} + 9900b^4c_{-1}^2 + 10668b^3b'c_{-1}^2 \\ &+ 2639b^2b'^2c_{-1}^2][3b'(24b + 13b')(15b' + 30bc_{-1} \\ &+ 13b'c_{-1})]^{-1} \end{aligned} \tag{A8a}$$

$$T^\theta{}_\theta = -\frac{4}{M^4}b' - T^t{}_t, \tag{A8b}$$

with c_{-1} determined in terms of b and b' by (A7e). Because of the appearance of the square root in (A7e) the dependence of the stress tensor components on b and

b' is not simply linear as in (4.15) or (4.16) of the known exact behavior on the ERN horizon. The numerical values of the coefficients for spin 0 and spin $1/2$ fields are also no better approximations to the numerical results than in the simpler solution considered in the text. As for that solution (4.14) there remains the freedom to adjust T^{uu} which still depends on a_0 and c_1, both of which are still free.

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