VALIDITY OF THE SEMICLASSICAL APPROXIMATION DURING THE PREHEATING PHASE OF CHAOTIC INFLATION

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The semiclassical approximation has frequently been used to describe the initial stage of particle production, often called preheating, which occurs after the inflationary epoch in chaotic models of inflation. During this phase backreaction effects from the produced particles on the inflaton field are significant, and one might be concerned about the validity of the semiclassical approximation, even though large backreaction effects are allowed if the inflaton field is coupled to a large number of quantum fields. A criterion is presented for the validity of the semiclassical approximation in this case and the question of whether this criterion is satisfied during preheating is addressed.

It is well known that during the preheating phase of chaotic inflation it is possible for a large amount of particle production to occur due to parametric amplification [1]. The backreaction of the particles on the inflaton field (ϕ) eventually causes its oscillations to damp and the particle production rate decreases. In the early stages of the process this is usually taken into account using the semiclassical approximation to compute the effects of the quantum fields (ψ) on the inflaton field. For example, in a model with a $g^2\phi^2\psi^2$ coupling, the equation for the inflaton field takes the form

$$\Box\phi - (m^2 + g^2\langle\psi^2\rangle)\phi = 0 \; . \tag{1}$$

Eventually scattering effects between the created particles (ignored by the semiclassical approximation) become important, but these should not be important during the early stages of the process. For earlier times it is usually assumed that the semiclassical approximation is valid. However, because of (i) the large amount of particle production that occurs and (ii) the large effects the particles have on the inflaton field, the semiclassical approximation is pushed harder in this situation than in most other situations where it is used, such as black hole evaporation [2]. Therefore, it is important to investigate the question of whether the semiclassical approximation is valid during the first part of the preheating phase of chaotic inflation, when parametric amplification is occurring and backreaction effects are significant.

There are different issues relating to the validity of the semiclassical approximation. One is that it can be obtained using a loop expansion of the effective action, which is in some sense an expansion in powers of $\hbar$. If there is only a small number of fields, then one expects that higher order terms in the expansion will be important if quantum effects are large. However, if there is a large number N of identical quantum fields, then an expansion can be obtained in inverse powers of N, with the result that the semiclassical approximation is the leading order in the expansion. In this case it should be possible to use the semiclassical approximation to determine backreaction effects, even when the quantum effects are large. However, there are higher order terms in the expansion which can become large when scattering effects from the produced particles (along with other effects) become important [3]. In the later stages of the preheating process in chaotic inflation such scattering effects are important [4], and thus, the semiclassical approximation breaks down.

There is a second issue regarding the validity of the semiclassical approximation which is related to the question of quantum fluctuations. The assumption made in using the semiclassical approximation is that quantum fluctuations about the mean value should be small. One way to address the question of whether they are small is to look at correlation functions. For example, for the above problem, one might look at the behavior of $\langle\psi^2(x)\psi^2(x')\rangle$. But there are problems with this quantity as it stands, including the existence of state dependent divergences. These have been identified for the two-point correlation function of the energy-momentum tensor in Ref. [5].

However, there is a natural way in which a two-point correlation function appears: in the linear response equations which are obtained when a solution to the semiclassical backreaction equations is perturbed. These equations can be obtained from a second variation of the effective action for the system, and no new types of divergences occur. This was illustrated in the case of the linear response equations for semiclassical gravity in Ref. [6]. A similar derivation for the inflaton field gives the general linear response equation [1]:

$$(\Box - m^2 - g^2\langle\psi^2\rangle)\delta\phi - (g^2\delta\langle\psi^2\rangle_{\rm SI} + g^2\delta\langle\psi^2\rangle_{\rm SD})\phi = 0\ , \tag{2}$$

$$\delta\langle\psi^2\rangle_{\rm SI} = -ig^2\int d^4x'\ \phi(x')\delta\phi(x')\theta(t-t')\langle[\psi^2(x),\psi^2(x')]\rangle\ . \tag{3}$$

Here the subscript "SI" refers to a variation that depends only on the state of the field before the variation and is independent of any variation in the state. The subscript "SD" denotes a variation in the state of the quantum field.

In Ref. [6] a criterion was proposed for the validity of the semiclassical approximation in gravity. An adaptation to the semiclassical approximation used in Eq. (1) is "the linear response equation for the inflaton field should have no solutions with finite non-singular initial data which grow without bound". It is important to note that this is a necessary but not a sufficient condition. It has two primary advantages. One is that it stays within the semiclassical approximation, so that it is not necessary to compute terms that have been neglected by the semiclassical approximation. The other is that no new types of divergences appear, and in particular, there are no state dependent divergences.

In what follows we investigate the validity of the semiclassical approximation during the preheating phase of chaotic inflation using the above model, which consists of a classical inflaton field ϕ with mass m, coupled to N identical massless quantum fields ψ. The coupling is of the form $g^2\phi^2\psi^2$. Full backreaction effects for this coupling have been investigated in detail in Refs. [4, 7–10] (although not all of these were in the context of the large N expansion). As mentioned above, using a large N expansion allows quantum effects to have a significant influence on the inflaton field.

[1] Note that state dependent variations were not discussed in Ref. [6].

To begin we need to be more specific about the solutions to Eq. (1) that we want to consider. In Ref. [10] we worked in a Minkowski spacetime background, a good approximation for the rapid damping phase, which occurs over timescales that are short compared to the expansion time of the universe [4] [2]. We considered only homogeneous solutions. We worked in the context of a large N expansion which, after rescaling the coupling constant g, results in effectively a single massless quantum field ψ coupled to the classical inflaton field ϕ. The mass of the inflaton field can be scaled out of the problem by letting

$$\bar{t} = mt \quad \text{and} \quad \bar{\phi} = \phi/m \;, \tag{4}$$

with similar changes of variable for other quantities that occur in the equations. See Ref. [10] for details. After dropping the "bars" one finds the following coupled set of equations:

$$\ddot{\phi} + (1 + g^2\langle\psi^2\rangle)\phi = 0 \;, \tag{5}$$

$$\langle\psi^2\rangle = \frac{1}{2\pi^2}\int_0^{\epsilon} dk\, k^2 \left(|f_k(t)|^2 - \frac{1}{2k}\right) + \frac{1}{2\pi^2}\int_{\epsilon}^{\infty} dk\, k^2 \left(|f_k(t)|^2 - \frac{1}{2k} + \frac{g^2\phi^2}{4k^3}\right) - \frac{g^2\phi^2}{8\pi^2}\left[1 - \log\left(\frac{2\epsilon}{M}\right)\right] \;, \tag{6}$$

$$\ddot{f}_k + (k^2 + g^2\phi^2) f_k = 0 \;. \tag{7}$$

The quantum state of the field can be obtained by choosing the starting values for the modes f_k. In Ref. [10] a fourth order adiabatic state was chosen using a WKB expansion for the modes of the form:

$$f_k(t) = \frac{1}{\sqrt{2W_k(t)}} \exp\left[-i\int_0^t W_k(t')dt'\right] \;. \tag{8}$$

Substitution into Eq. (6) gives an equation for W_k which can be solved iteratively with the lowest order solution given by $W_k = k$. The actual state chosen for the numerical integrations that were shown in Ref. [10], is given by

$$W_k^{-1}(0) = \frac{1}{(k^2 + g^2\phi^2(0))^{1/2}} + \frac{g^2[\dot{\phi}^2(0) + \phi(0)\ddot{\phi}(0)]}{4(k^2 + g^2\phi^2(0))^{5/2}} \;. \tag{9}$$

In Fig. 1 we show two plots from Ref. [10] for $g = 10^{-3}$, two different initial values for ϕ, and fourth order adiabatic vacuum states appropriate for these initial conditions. There is no rapid

[2] As discussed in Ref. [10], there are actually two rapid damping phases which were always observed. It is possible that the Minkowsky spacetime approximation is not very good during the time between them since that time can be relatively long compared to the rapid damping timescale.

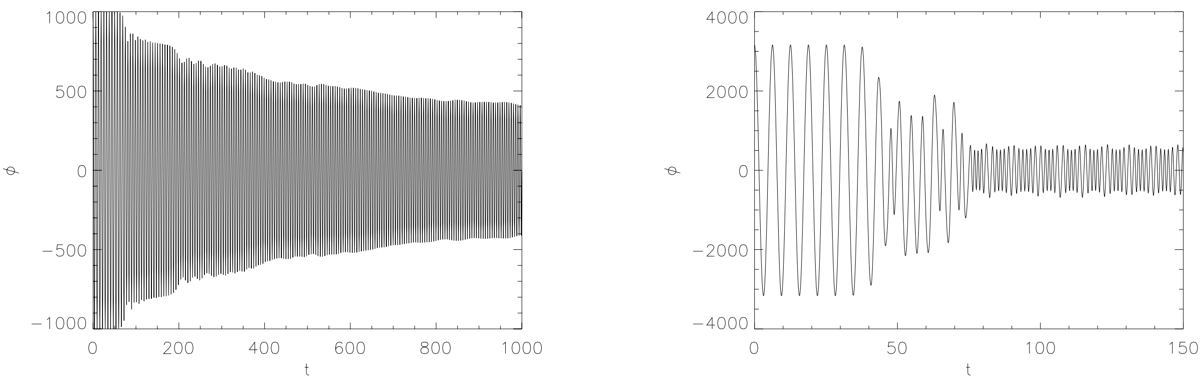

FIG. 1: These plots from Ref. [10] show the time evolution of the inflaton field. In both cases $g = 10^{-3}$ and $\dot{\phi}(0) = 0$. For the plot on the left $g^2\phi^2(0) = 1$ and for the plot on the right $g^2\phi^2(0) = 10$.

damping phase for the figure on the left, but there is for the figure on the right. As discussed in Ref. [10], it was found in all the cases considered that, if $\dot{\phi}(0) = 0$, then rapid damping occurs for $g^2\phi^2(0) \gtrsim 2$.

There are at least two ways that one can derive the linear response equation. One is by varying the closed-time-path (CTP) effective action twice. The other is by directly perturbing the equation for the inflaton field, along with the equations for the modes of the quantum field. The latter method, while less elegant, gives some important insights into the linear response equation, and also leads to a simple way to find approximate solutions to it. We shall only sketch it here; the details will be given elsewhere.

One begins by perturbing both the backreaction equation for the inflaton field and the mode equation in the usual way, keeping only quantities that are first order in the perturbations, either $\delta\phi$ or δf_k. The mode equation can then be solved in terms of an integral over a one-dimensional Green function (built with the solutions to the zeroth order mode equation) with the result:

$$\delta f_k = A_k f_k + B_k f_k^* + 2gi \int_0^t dt' \; \phi(t')\delta\phi(t') f_k(t')[f_k^*(t) f_k(t') - f_k(t) f_k^*(t')] \; . \qquad (10)$$

The coefficients A_k and B_k are fixed by the initial values of δf_k and its first derivative. If either or both are non-zero, then there is a change of quantum state. In fact, this will always occur if the original state is a second order (or higher) adiabatic state and $\delta\phi \neq 0$, because the initial value for δf_k will depend upon the initial value of $\delta\phi$. Conversely, even if $\delta\phi(0) = \delta\dot{\phi}(0) = 0$, a change in state will generate a non-zero $\delta\phi$ at later times through the linear response equation. The term $\delta\langle\psi^2\rangle_{SD}$ in the linear response equation (2) is composed of those terms which depend upon A_k and

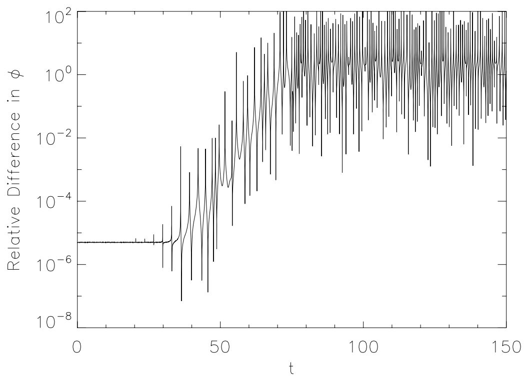

FIG. 2: This plot from Ref. [10] shows the time evolution of $|\delta\phi/\phi_1|$ for the case $g = 10^{-3}$, $\dot{\phi}(0) = \delta\dot{\phi}(0) = 0$, $g^2\phi_1^2(0) = 10$, and $\delta\phi = 10^{-5}\,\phi_1(0)$.

B_k, while the term $\delta\langle\psi^2\rangle_{SI}$ is composed of the terms which do not depend on A_k and B_k. For the fourth order adiabatic states used in Ref. [10], we find that $A_k = 0$ to linear order. An explicit expression for B_k can easily be obtained but we will not display it here.

Since we have a numerical code that solves the original backreaction equation (1), it is easy to generate approximate solutions to the linear response equation (2). One simply takes two solutions, ϕ_1 and ϕ_2, which have nearly the same values at the initial time $t = 0$, and evolves them numerically in time. If we define the difference between the solutions to be $\delta\phi \equiv \phi_2 - \phi_1$, then $\delta\phi$ satisfies the exact equation:

$$\delta\ddot{\phi} + (1 + g^2\langle\psi^2\rangle_1)\delta\phi + g^2(\langle\psi^2\rangle_2 - \langle\psi^2\rangle_1)(\phi_1 + \delta\phi)\ . \tag{11}$$

The linear response equation (2) is in this case:

$$\delta\ddot{\phi} + (1 + g^2\langle\psi^2\rangle_1)\delta\phi + g^2(\delta\langle\psi^2\rangle_{\rm SI} + g^2\delta\langle\psi^2\rangle_{\rm SD})\phi_1 = 0\ . \tag{12}$$

Note that the first term after $\delta\ddot{\phi}$ is exactly the same as in the linear response equation (2). Thus, the exact $\delta\phi$ which is a solution to Eq. (11) is also an approximate solution to Eq. (12) so long as

$$\begin{aligned} &|\frac{\delta\phi}{\phi_1}| \ll 1\ , \\ &\delta\langle\psi^2\rangle_{\rm SI} + g^2\delta\langle\psi^2\rangle_{\rm SD} \approx \langle\psi^2\rangle_2 - \langle\psi^2\rangle_1\ . \end{aligned} \tag{13}$$

In Fig. 2 a plot from Ref. [10] is shown which displays a numerical solution to Eq. (11). At early times not much happens because the mass term in Eq. (12) dominates. Once parametric amplification has made the quantum effects large enough, the perturbation begins to grow exponentially.

For $t \gtrsim 70$, it was found that the terms in the two equations (11) and (12) are not similar in size, and $\delta\phi$ is no longer an approximate solution to the linear response equation (12). Note that $\delta\phi$ grows exponentially by about five orders of magnitude at the same time that the inflaton field ϕ_1 goes through nine oscillations. A detailed numerical analysis shows that, on average and during this time, the largest non-derivative term in the linear response equation is the term containing $\langle\psi^2\rangle_{\rm SD}$.

As noted in Ref. [10], we find that in the cases studied, $\delta\phi$ grows exponentially during the period of rapid damping (which is actually the period between the first rapid damping and the second). We find here that, on average and during this period, the largest term in the linear response equation is that containing $\langle[\psi^2(x), \psi^2(x')]\rangle$. This provides strong evidence that quantum fluctuations are large during, at least part of, the preheating phase of chaotic inflation if a period of rapid damping occurs. Therefore, the semiclassical approximation may not be valid during the rapid damping phase in cases where it occurs.

In the process of carrying out this work certain technical difficulties relating to the derivation and solution of the linear response equations have been worked out. In that sense, this work paves the way for studies of the validity of the semiclassical approximation in gravity in both cosmological and black hole spacetimes.

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