Sensitivity of low-rank matrix approximation and recovery

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Consider the manifold of m×n matrices of rank r:

$$\mathcal{M}_r := \{ Y \in \mathbb{R}^{m \times n} \mid \operatorname{rank}(Y) = r \}$$

The Frobenius-norm for $A \in \mathbb{R}^{m \times n}$ is

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Low-rank-matrix

approximation

Let us consider the following optimization problem: for input $A \in \mathbb{R}^{m \times n}$

$$(\text{APPROX}) \quad \underset{Y \in M_r}{\operatorname{argmin}} \ \frac{1}{2} \| A - Y \|_F^2$$

The solution of (APPROX) is determined by the singular value decomposition of A.

Research-question: We want to understand the sensitivity of Y with respect to perturbations in A.

This sensitivity is measured by a condition number:

$$\kappa_{APPROX}(A) := \lim_{\varepsilon \to 0} \sup_{\|\Delta A\| < \varepsilon} \frac{\| Y(A) - Y(A+\Delta A)\|_F}{\|\Delta A\|_F}$$

(and $\kappa_{APPROX}(A) := \infty$, if $\sigma_r(A) = \sigma_{r+1}(A)$).

Interpretation:

For small $\|\Delta A\|_F$: $\| Y(A) - Y(A+\Delta A)\|_F \lesssim \kappa_{APPROX}(A) \cdot \|\Delta A\|_F$

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$\kappa_{APPROX}(A)$ large

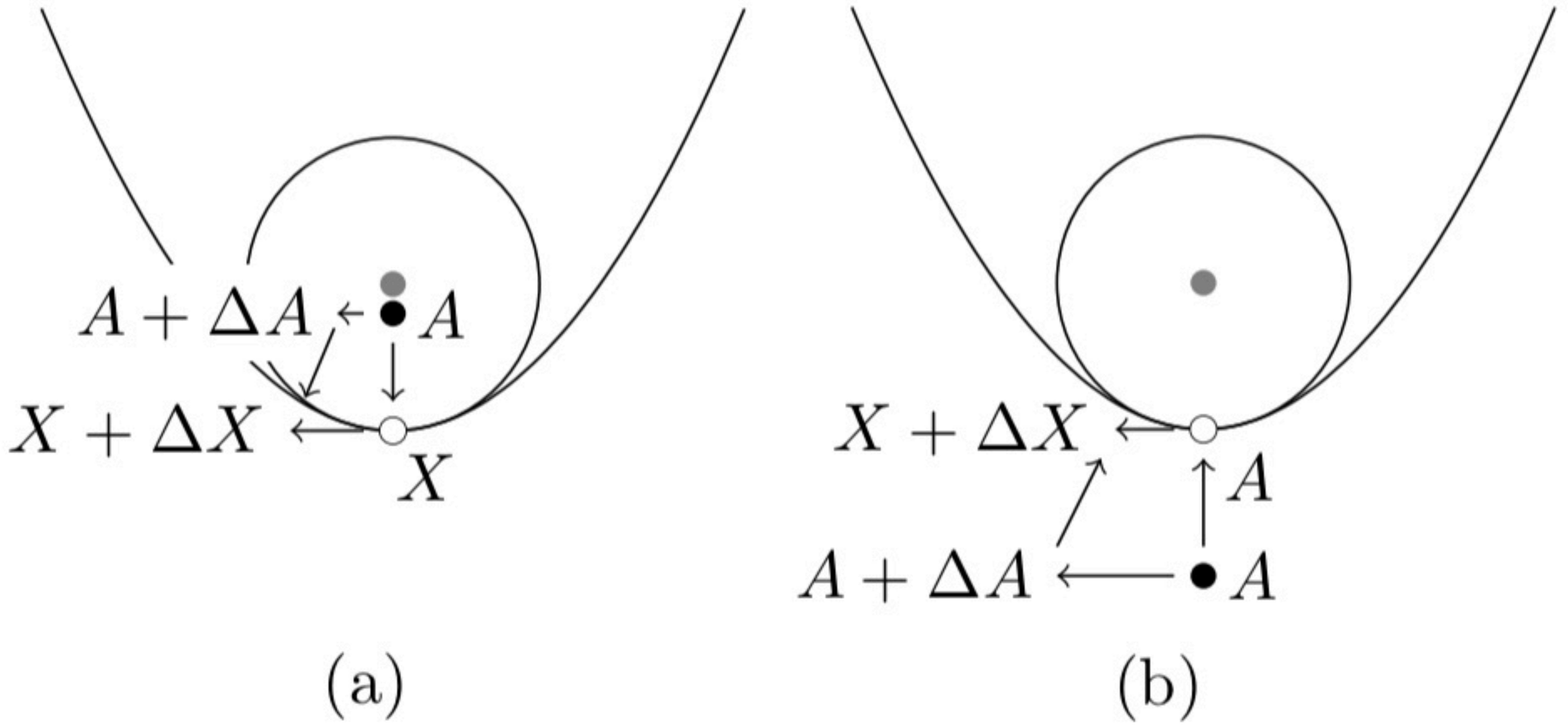
=

high sensitivity of $Y(A)$ wrt errors in A

=

BAD!

The sensitivity of $X = \underset{Y \in \mathcal{M}_r}{\operatorname{argmin}} \frac{1}{2}\|A - Y\|_F^2$ with respect to perturbations in A depends on the curvature of $\mathcal{M}_r$ at X:



Theorem (B., Vannieuwenhoven, 2021)

Let $A \in \mathbb{R}^{m,n}$ and $\sigma_1 \geq \dots \geq \sigma_{\min(m,n)}$ be the singular values of A. Then:

$$\kappa_{\text{APPROX}}(A) = \frac{1}{1 - \sigma_{r+1}/\sigma_r}$$

Low - rank- matrix

recovery

Next setting: Let $A \in \mathbb{R}^{\ell}$, when $\ell < m \cdot n$.

Consider the recovery problem

$$\text{(RECOVERY)} \qquad \operatorname*{argmin}_{Y \in \mathcal{M}_r} \frac{1}{2} \| A - L(Y) \|^2,$$

where $L : \mathbb{R}^{m \times n} \to \mathbb{R}^{\ell}$ is affine linear.

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Consider the recovery problem

$$(\text{RECOVERY}) \quad \underset{Y \in \mathcal{M}_r}{\operatorname{argmin}} \ \frac{1}{2} \| A - L(Y) \|^2 ,$$

(euclidean norm on $\mathbb{R}^{\ell}$)

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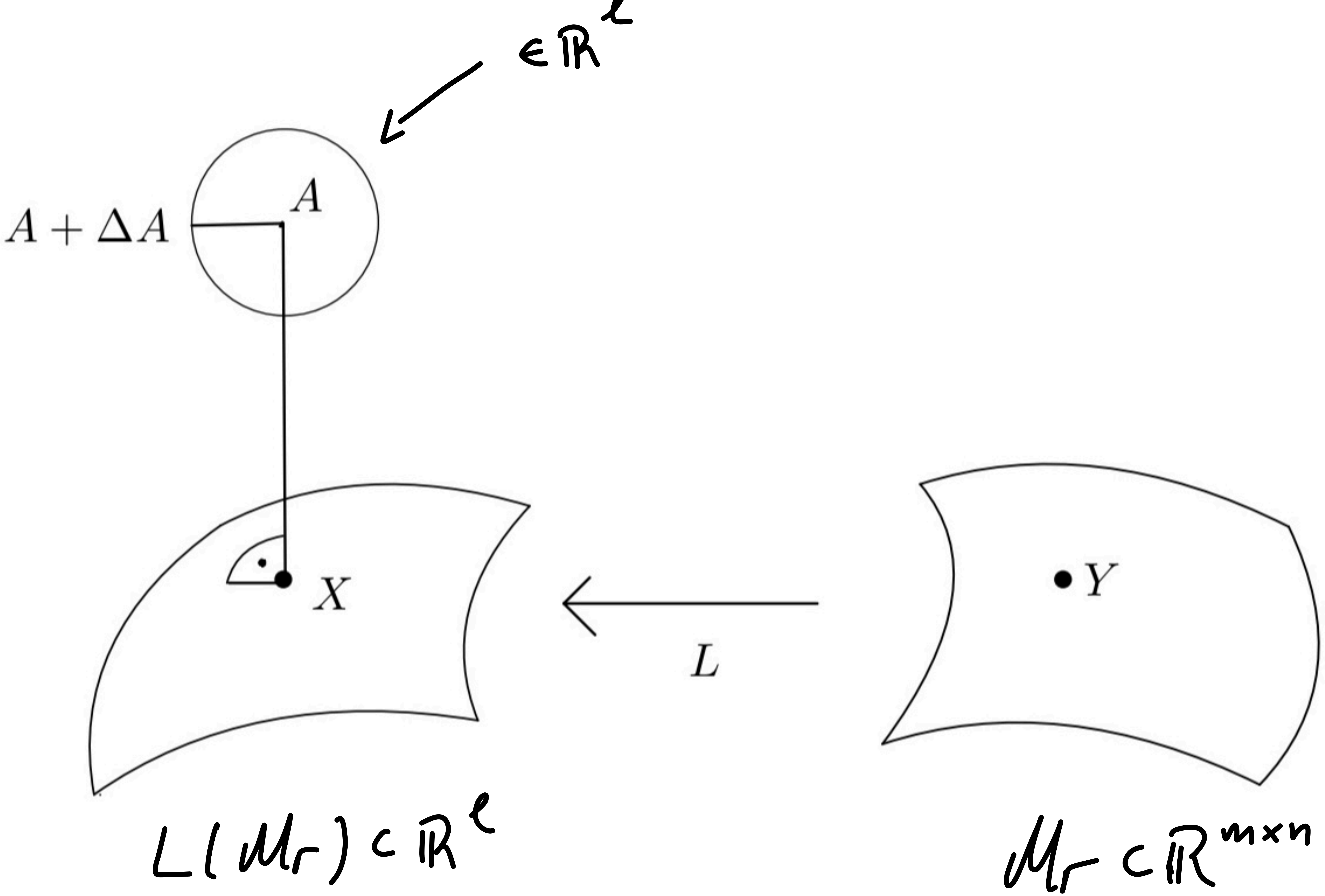
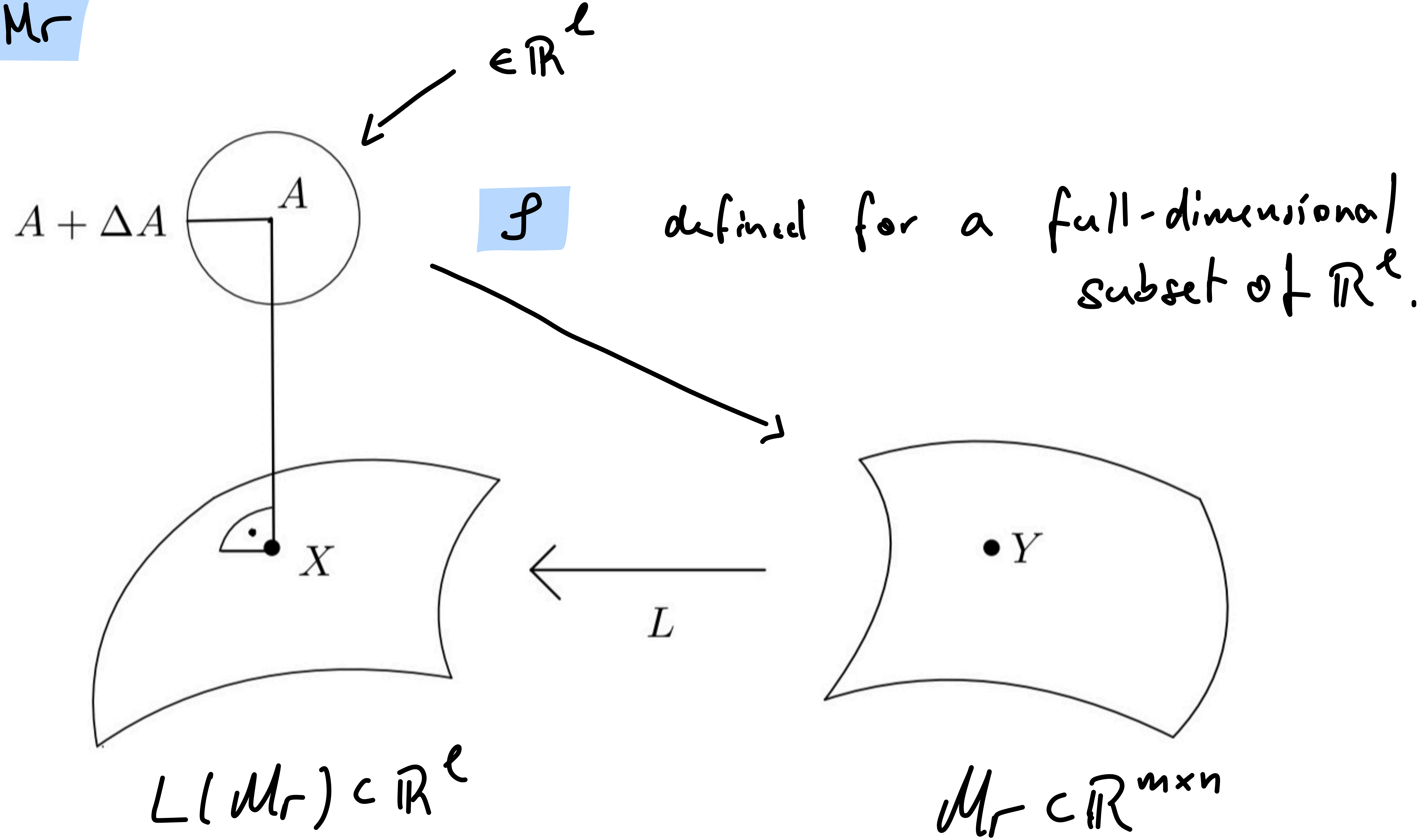


∈ ℝ^ℓ
A
A + ΔA
X
Y
L
L(M_r) ⊂ ℝ^ℓ
M_r ⊂ ℝ^{m×n}

If $\ell > \dim M_r$



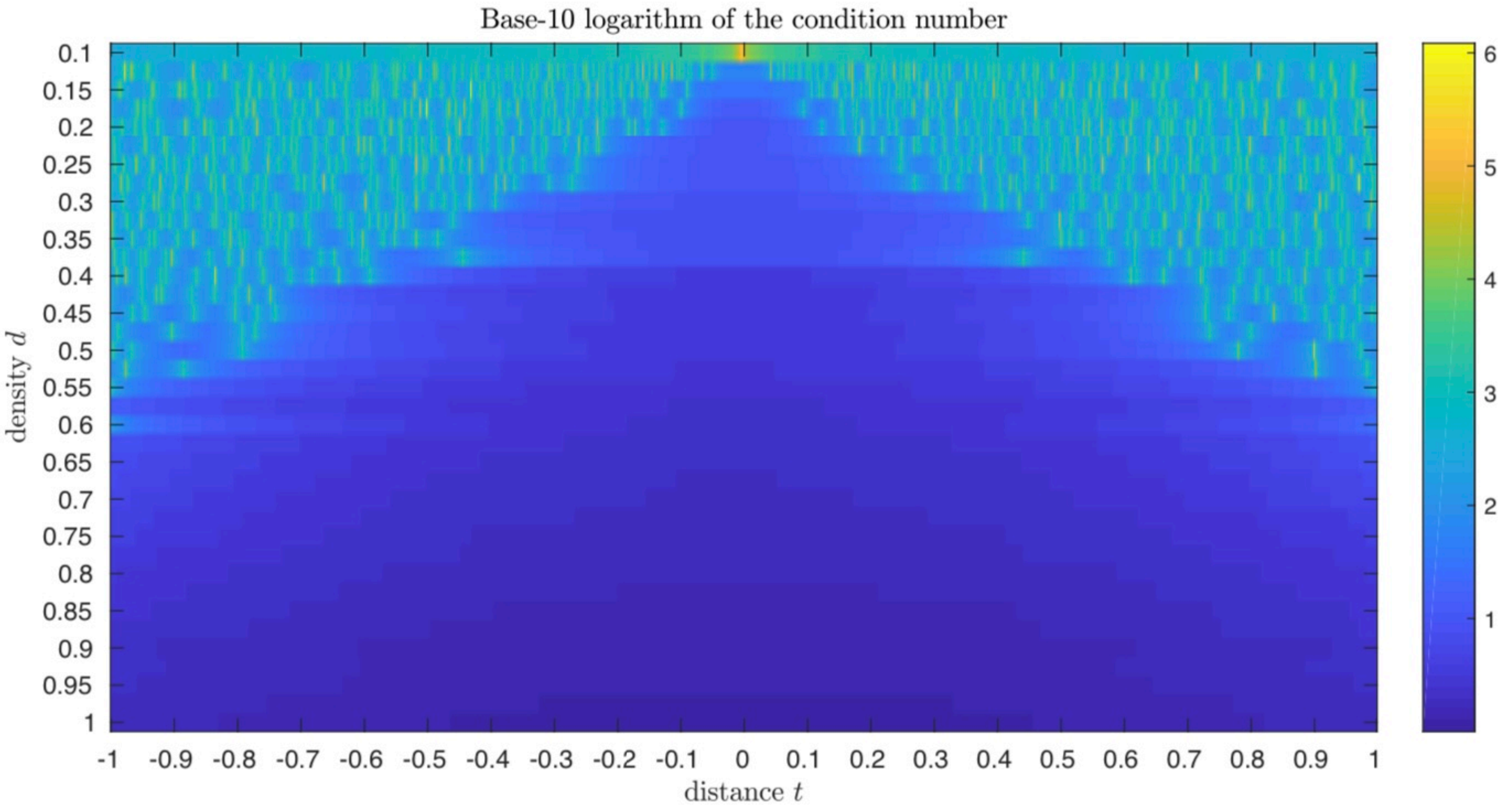
$\in \mathbb{R}^\ell$
$A + \Delta A$
A
f
defined for a full-dimensional subset of $\mathbb{R}^\ell$.
X
L
Y
$L(\mathcal{M}_r) \subset \mathbb{R}^\ell$
$\mathcal{M}_r \subset \mathbb{R}^{m \times n}$

The condition number of low-rank-matrix recovery is

$$\kappa_{RECOVERY}(A) = \lim_{\varepsilon \to 0} \sup_{\|\Delta A\| < \varepsilon} \frac{\| f(A+\Delta A) - f(A) \|_F}{\|\Delta A\|},$$

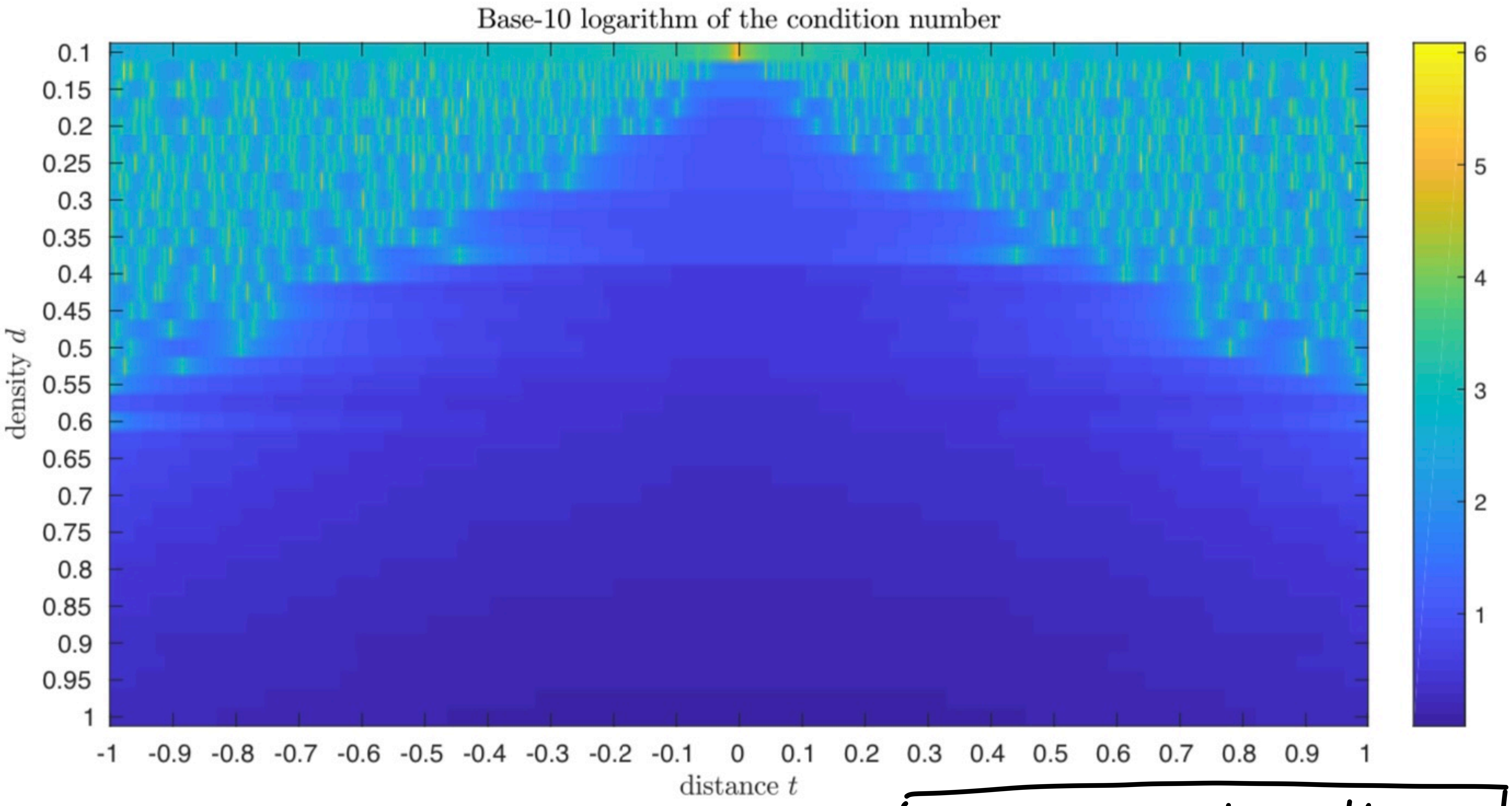
where $f(A) = \underset{Y \in \mathcal{M}_r}{\operatorname{argmin}} \frac{1}{2} \| A - L(Y) \|^2$

$d = \frac{\ell}{m \cdot n}$



$m = n = 50$, $r = 2$. $t = \|A - L(Y)\|$

$d = \frac{\ell}{m \cdot n}$



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→ more observations = less sensitivity

Take away:

1. Sensitivity of low-rank approximation depends on the singular value gap.

2. In low-rank matrix recovery more observations decrease the sensitivity.

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THANK YOU FOR YOUR ATTENTION!