On the Monotonicity of the Option-Value/Risk Relation

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Abstract
The belief that, regardless of strike price, option values increase monotonically as risk increases is widely held. This note shows that this “monotonicity hypothesis” is true only in very limited situations and is not a general result. I show that a necessary condition for the monotonicity hypothesis to obtain is that the distribution of the underlying asset’s value at expiry can be completely characterized by the mean and a single risk measure. This is true for 2-parameter parametric families of distributions such as the log-normal distribution assumed in Black and Scholes’ and Merton’s original works on the subject. However, in general no two statistics can suffice to uniquely describe a distribution. Therefore whether an increase in risk increases the value of an option depends on the strike price and how the change in risk changes the underlying distribution.

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The Monotonicity of the Option-Value/Risk Relation

Every introductory investments and derivatives text teaches that the value of an option increases monotonically as the risk of the underlying asset increases. This relation is widely assumed to be true, usually without qualification, among academics and practitioners. The hypothesis appears in the real options approach to capital budgeting.

In the literatures on corporate governance and bank capital regulation, the monotonicity hypothesis underpins moral hazard arguments that equity holders, who hold a call option on the assets of the firm, prefer riskier investments that increase the value of this option, tempting owner/managers to engage in excessive risk taking at the expense of the bondholders and deposit insurers.

The monotonicity hypothesis originated in the work of Black and Scholes (1972) and Merton (1973). Within the context of the distributional assumptions and European options studied in these papers the relation does indeed hold true. Thus, if stock prices follow a geometric Brownian motion, it is indeed true that European option prices (should theoretically) increase as the variance of the price process increases. In entering common knowledge, the conditional nature of this result has been lost. It is, however, relatively simple to construct individual counter-examples to the monotonicity hypothesis for European options (e.g., Jaganathan, 1984). That alone suffices to establish that the monotonicity hypothesis is not universally true.

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1 The statement in Hull (1993, p. 153), “The values of both calls and puts therefore increase as volatility increases.” is representative of the norm. The statement occurs in a section that makes no mention of assumptions concerning the distribution of returns, though it does restrict the discussion to European and American options.


3 Consider the following example presented in a Harrison-Kreps no-arbitrage pricing framework: Let the risk-free rate be 5 percent. Two assets, A and B, both have current values of 1. The payoffs and variances are given in the following table along with the values of call options with strikes of 1 and 2 units.

<table>
<thead>
<tr>
<th>State</th>
<th>Payoffs</th>
<th>( \sigma^2 )</th>
<th>C(K=1)</th>
<th>C(K=2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>Medium</td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td>Probability</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>Asset A</td>
<td>2.041</td>
<td>0.559</td>
<td>0.500</td>
<td>0.491</td>
</tr>
<tr>
<td>Asset B</td>
<td>2.020</td>
<td>1.130</td>
<td>0.000</td>
<td>0.683</td>
</tr>
</tbody>
</table>

The expected payoff of each asset is 1.05, consistent with risk-neutral pricing. The value of C(K=1) is increasing in \( \sigma^2 \) as the monotonicity hypothesis suggests. However, C(K=2) is decreasing in \( \sigma^2 \).
Outside the European option context, the monotonicity hypothesis is not necessarily true, even when underlying returns are log-normally distributed.\textsuperscript{4} Few contingent claims, other than some traded options, are simple European puts and calls on non-dividend-paying underlying assets. Most corporate budgeting, regulatory and risk management problems involve multiple periods, boundary conditions, uncertain intermediate payouts, and the potential for adaptive adjustments of portfolio characteristics. These complications can create conditions where we either know the monotonicity hypothesis does not obtain or where we have no idea whether it does or not does not.

**Literature Review of Sufficient Conditions**

A number of papers have established sufficient conditions under which the monotonicity hypothesis will hold for plain-vanilla European put and call options. Black-Scholes (1972) and Merton (1973) assume that the underlying asset follows a log-normal diffusion process:

$$\frac{dS}{S} = \alpha dt + \sigma(S,t)dz$$

and pays no dividends. In the Black-Scholes model, the volatility term is constant, $\sigma(S,t) = \sigma$. In the Merton model the volatility term can vary as a deterministic function of time, $\sigma(S,t) = \sigma(t)$. In both cases the monotonicity hypothesis will obtain. Bergman, Grundy, and Weiner (1996) show that the monotonicity hypothesis also holds for the general class of univariate diffusion processes even if volatility of the stochastic processes being compared depends on the level of the underlying asset as long as $\sigma^*(s,t) \geq \sigma(s,t) \forall s,t$ and $\sigma^*(s,t) > \sigma(s,t)$ for some $s$ and $t$.\textsuperscript{5}


\textsuperscript{4} Barrier options are a case in point. When the underlying asset’s price is close to the barrier, the barrier option’s value is inversely related to risk. See Merton (1978), Ritchken, Thomson, DeGennaro and Li (1993), and Geske and Shastri (1981) for other examples.

\textsuperscript{5} Geske and Shastri (1981) and Bergman, Grundy and Weiner (1996) study other commonly held beliefs: 1) that option values change monotonically with the value of the underlying asset; 2) that option values increase monotonically with the time to expiry; 3) that option values are convex functions of the value of the underlying. All of these hypotheses can be violated when one moves away from the Black-Scholes-Merton framework.
in a two-period context that if the terminal-value distributions of two underlying assets differ in risk in the Rothschild-Stiglitz (1970) sense of mean-preserving spreads, then otherwise-similar options on the riskier asset will be more valuable. Thus, the ability to order asset return distributions in a Rothschild-Stiglitz risk ordering is also a sufficient condition for the monotonicity hypothesis to obtain. However, the Rothschild-Stiglitz definition of risk is very restrictive. In general, two distributions will not differ by mean-preserving spreads, and thus cannot be ranked on the basis of their “risk” in the Rothschild-Stiglitz sense.

Each of these papers establishes a sufficient condition under which the monotonicity hypothesis will be true. However, these sufficient conditions need not obtain in general, and so provide only special cases. Other distributional assumptions, for example those in footnote 3, can produce violations of the hypothesis. This ambiguity leads us to ask whether the conditions under which the hypothesis is false are unusual, or whether the conditions where it is true are the unusual cases. To answer this question we can look at a necessary condition for the hypothesis to be true. By addressing the question of necessary conditions for the monotonicity hypothesis to obtain we can gain a broader understanding of how widely applicable it might be. If the necessary conditions obtain in most situations, and the exceptions are of little practical importance, then the common wisdom that option prices increase with risk can be accepted (with the qualifier “usually”). If however, the necessary conditions for the hypothesis to obtain are not usually met, or cannot reasonably be claimed to be met in a particular instance, then the monotonicity hypothesis looses its generality and usefulness.

I proceed by assuming that the monotonicity hypothesis is true and then derive an implication of this assumption. I show that, in most instances, this implication is counterfactual. Since the implication of the hypothesis is not generally true, the hypothesis itself cannot be generally true. In other words, we proceed using the syllogism

\[(A \rightarrow B) \&\ not(B) \rightarrow not(A)\]

I will focus on what the monotonicity hypothesis, the \(A\), implies about the distribution of asset prices at option expiry, the \(B\).
Step 1: Drawing the Implication

Rather than focusing on hypothesized stochastic processes for the underlying asset or utilizing the Harrison-Kreps framework, we will employ an alternative framework for pricing European options developed in Cox-Ross (1976). This framework is consistent with the other frameworks, but is more general than those based on specific stochastic processes. Cox and Ross showed that the value of an option is its discounted expected payoff, where the expectation is taken with respect to the risk-neutral distribution of the value of the underlying asset at option expiration. For a call option this means

$$C(S_t; K, T) = e^{-r(T-t)} \int_{-\infty}^{\infty} (x - K) f(x) dx$$  (1.1)

where $f(\cdot)$ is the risk-neutral density function for the terminal, time-$T$, value of $S_T$ (henceforth “the terminal density”). To guarantee that $f(\cdot)$ is a proper distribution

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

and

$$0 \leq f(x) \leq 1, \forall x \in (-\infty, \infty).$$

Furthermore, under risk-neutrality the discounted expected value of the underlying asset must equal its current price:

$$E(S_T; S_t) = \int_{-\infty}^{\infty} x f(x) dx = S_t e^{r(T-t)}$$  (1.2)

to ensure that the drifts under the risk-neutral distribution equal the risk-free rate, $r$.

The monotonicity hypothesis is usually stated loosely as “increasing risk increases the value of an option” or “an option on a riskier asset will be more valuable.” In either case it is a statement about options on different distributions—either different underlying assets or the same underlying asset before and after the distribution has changed. When analyzing the effects of risk on the value of options on underlying assets with different terminal densities, the non-risk-related characteristics—moneyness, time-to-expiry, and
types of option—must be equivalent. With this proviso, let us state the monotonicity hypothesis as:

\[
\text{Risk}(S_T) < \text{Risk}(S_T^*) \Rightarrow C(S_T; K, S_T) \leq C(S_T^*; K, S_T^*) \quad \forall K, S_T, t, T
\]  

(1.3)

where \( K \) is the common strike price, \( t \) and \( T \) are the current time and option expiry respectively, and \( S_T \) and \( S_T^* \) are the current values of the two underlying assets, \( S_T = S_T^* \) per the preceding comparability condition, and \( S_T \) and \( S_T^* \) are the (random) values of the two underlying assets (or the same underlying asset before and after the terminal distribution changed) at the expiration of the option. “Risk” is, for the moment, undefined. The monotonicity hypothesis is unconditional and therefore equation (1.3) must hold for all strikes, \( K \). Thus

\[
C(S_T; K, S_T) > C(S_T^*; K, S_T^*) \text{ for some } K \Rightarrow C(S_T; K, S_T) \geq C(S_T^*; K, S_T^*) \quad \forall K.
\]

In other words, if the monotonicity hypothesis is true, option-value ordering cannot be strike-dependent.

The first thing to note is that the monotonicity hypothesis presumes a mapping from the terminal distributions of the underlying assets to an orderable, that is univariate, measure termed “Risk.” Otherwise the hypothesis is vacuous. In the Black-Scholes-Merton framework the volatility (variance) of the stochastic process provides this measure. In the Jaganathan analysis, successively applying mean-preserving spreads to any base density will produce an ordered set of increasingly riskier terminal densities.

Suppose that the class of admissible asset terminal-value distributions admits the possibility that there exist a pair of underlying assets, \( S_T \) and \( S_T^* \), with associated

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6 Obviously a deep in-the-money option will be worth more than a deep out-of-the-money option on a slightly riskier underlying asset. Equivalence of moneyness can be achieved by assuming, for purposes of this argument, the current value of the underlying and the strikes are the same.

7 It is important to note that this equality holds only at the time the option values are being compared. A time progresses from \( t \) to \( T \) the price paths of the underlying assets can, of course, diverge.

8 It is common practice to identify the underlying asset’s “risk” with the volatility or variance of the underlying stochastic process or the variance of the distribution of asset values at the time of option expiration.

9 The example in footnote 3 violates this condition. Because options on A are sometimes more valuable than options on B and sometimes less, it is impossible, in this case, for any risk measure to be consistent with the hypothesis that riskier assets leads to higher option values. Thus, for this distribution of payoffs, the monotonicity hypothesis cannot hold for any measure of risk.
terminal densities, $f(S_t)$ and $f^*(S^*_t)$, for which $\text{Risk}(S_t) = \text{Risk}(S^*_t)$.$^{10}$ Under the monotonicity hypothesis this would imply that

$$C(S_t; K, S_t) = C(S^*_t; K, S^*_t) \quad \forall K$$

and therefore that

$$\int_{K}^{\infty} (x - K) f(x) dx = \int_{K}^{\infty} (x - K) f^*(x) dx \quad \forall K. \quad (1.4)$$

Following Breeden and Litzenberger (1978) and differentiating (1.4) twice with respect to $K$ results in

$$f(K) = f^*(K) \quad \forall K.$$ 

In other words, if the monotonicity hypothesis is true, equal “risk” implies that the two terminal densities must be identical. Of course, the converse is always true: identical terminal densities will have identical risks for any conceivable measure of risk.

Thus, the monotonicity hypothesis can only be true if the mapping from the space of admissible terminal density functions to the risk measure is one-to-one.$^{11}$ If the measure of risk we are using is not a one-to-one mapping, then two different densities could have the same risk and yet have different option values for some strikes. This would violate the monotonicity hypothesis. Thus:

_A necessary condition for the monotonicity hypothesis to obtain is the existence of a one-to-one mapping from the space of admissible terminal densities to an orderable univariate measure we call risk._

This establishes the first premise, $A \rightarrow B$, of our syllogism, $B$ being “there existences a one-to-one mapping between the space of terminal price densities and a univariate measure of risk.”

$^{10}$ This assumption says no more than that asset prices/returns are drawn from a sufficiently general set of distributions so as to admit the possibility that two assets may have the same risk. To assert that no two assets can have the same risk imposes a rather strong restriction on asset returns. If we are comparing options on a finite set of specific assets, rather than general classes of distributions, we are back in the realm of special cases. While the monotonicity hypothesis may or may not hold for particular subsets of assets, that tells us nothing about the hypothesis in general. After all, the hypothesis is not of the form “there exists a set of assets for which equation (1.3) holds.”

$^{11}$ Technically, to be one-to-one, each possible value of the risk measure must correspond to an admissible density. We can achieve this by limiting the admissible risk measures to values (or sets) defined by the admissible density functions.
Step 2: Is the Implication True?

It is well known that, in general, a density function cannot be uniquely identified by any finite set of moments, order statistics, or other numbers. It is only in cases where the admissible densities are restricted to known discrete distributions or parametric forms that a finite set of numbers will be sufficient to uniquely identify the densities. The monotonicity hypothesis requires that the admissible densities be uniquely identifiable by two numbers (“risk” and mean). In the continuous-distribution case this is true only for 2-parameter parametric families of densities. Even then it is necessary that the set of admissible densities be restricted to a single 2-parameter class.

Many of the stochastic processes frequently assumed in the option pricing literature, including the ubiquitous log-normal diffusion, do indeed produce terminal densities that have this necessary property. However, it is important to remember that these stochastic process assumptions are usually made for reasons of tractability, tradition, or habit. The validity of these assumptions is seldom tested directly. In the more conceptual literature of real options explicit assumptions are rarely made about asset distributions, though the conclusions necessarily rely on implicit assumptions, as we have just shown.

Merton himself admits that log-normality is a crude approximation of stock return distributions (Merton, 1990, p. 59). The well-known volatility smile is further evidence that the log-normal distributional assumption underlying the Black-Scholes-Merton framework is frequently counter-factual, at least for stock returns. Cross-sections of option prices appear to embed complicated empirical terminal densities. The literature on estimating risk-neutral densities directly from cross-sections of option prices (see, for example, Bliss and Panigirtzoglou, 2002) indicates that the simple log-normal model does not adequately describe the underlying processes implicit in equity or interest rate options prices. For other assets, for instance loan portfolios with their bounded returns, the log-normality assumption is clearly untenable on a priori grounds.

This condition, that distributions cannot be uniquely characterized by two statistics, mean and risk, constitutes the second premise of our syllogism, “not(B),” though to be precise, I am not arguing that “B” is always false; I am arguing that “B” is
not generally true. However, since the monotonicity hypothesis does not say “sometimes option value increases in risk,” it is sufficient to argue that “\( \neg(B) \)” is possible and indeed the usually true.

**Conclusion**

Having established the two premises of the syllogism, it follows that, the monotonicity hypothesis, as a general proposition, is false, that is: \( \neg(A) \). Thus, whether or not an option value increases when the risk of the underlying asset increases depends on how the entire distribution of terminal prices changes and on the strike price of the option. This is not to say that for some risk-increasing changes in distributions all (or some) option prices might not increase. We deny that for all risk-increasing changes in distributions, by whatever univariate (or finite-dimensional) statistic risk is measured, all option prices must increase.

I have shown that, in general, the monotonicity hypothesis requires tight restrictions on the terminal densities for underlying assets and thus on the admissible stochastic processes—conditions that were fortuitously met (by assumption) in the original Black-Scholes (1972) and Merton (1973) studies. Unless one is willing to assume that all underlying asset distributions come from the same 2-parameter parametric family or that all underlying asset distributions differ only by mean-preserving spreads, one cannot conclude that higher risk invariably implies higher European option value. There is usually little reason, theoretically or empirical, to make such an assumption when considering real-world options, even if for some other applications, such as pricing, it is a tractable approximation to reality.

Thus, the monotonicity hypothesis should not be taken for granted. Whether it applies in a particular instance is an empirical question that must be established on a case-by-case basis before the hypothesis can be invoked. As an unqualified, universal, qualitative result, the monotonicity hypothesis is false.
Bibliography


Hull, John C., 1993, Options, Futures, and Other Derivatives Securities 2e, Prentice-Hall, New Jersey.


