Toward Astrophysical Black-Hole Binaries

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Abstract

A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then a complete set of boundary conditions for all initial data variables can be developed. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data.


Collaborators: Harald Pfeiffer & Saul Teukolsky (Cornell)
Motivation

- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, . . .
- Available computed waveforms should increase chance of detecting collision events.

Why Quasi-Equilibrium?

- General Relativity doesn’t permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.
- Quasi-equilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

– Greg Cook – (WFU Physics)
**The 3 + 1 Decomposition**

Lapse: $\alpha$  
Spatial metric: $\gamma_{ij}$  
Shift vector: $\beta^i$  
Extrinsic Curvature: $K_{ij}$  
Time vector: $t^\mu = \alpha n^\mu + \beta^\mu$

\[
ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)
\]

\[
\gamma_{\mu\nu} = g_{\mu\nu} + n_{\mu} n_{\nu} \quad K_{\mu\nu} = -\frac{1}{2} \gamma_{\mu}^{\alpha} \gamma_{\nu}^{\beta} L n g_{\alpha\beta}
\]

**Constraint equations**

\[
\bar{R} + K^2 - K_{ij} K^{ij} = 16\pi \rho
\]

\[
\bar{\nabla}_j \left( K^{ij} - \gamma^{ij} K \right) = 8\pi j^i
\]

- Greg Cook – (WFU Physics)

**Evolution equations**

\[
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i
\]

\[
\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[ \bar{R}_{ij} - 2K_{i\ell}K^{\ell}_j + KK_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij} (S - \rho) \right] + \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell} \bar{\nabla}_j \beta^\ell + K_{j\ell} \bar{\nabla}_i \beta^\ell
\]

\[
S_{\mu\nu} \equiv \gamma_{\mu}^{\alpha} \gamma_{\nu}^{\beta} T_{\alpha\beta}
\]

\[
j_{\mu} \equiv -\gamma_{\nu}^{\rho} n^{\alpha} T_{\nu\alpha}
\]

\[
\rho \equiv n^{\mu} n^{\nu} T_{\mu\nu}
\]

\[
T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu} j_{\nu)} + n_{\mu} n_{\nu} \rho
\]
Degrees of Freedom

**Kinematical variables**
- Lapse \( \alpha \): 1 degree of freedom
- Shift \( \beta^i \): 3 degrees of freedom

**Initial-data variables**
- Metric \( \gamma_{ij} \): 6 degrees of freedom
- Extrinsic curvature \( K_{ij} \): 6 degrees of freedom

Decomposition of initial-data variables[5] \( (\text{Conformal TT decomp}) \)

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}
\]

\[
\begin{align*}
\psi : & \quad \text{1 constrained DOF} \\
\tilde{\gamma}_{ij} : & \quad \text{3 spatial gauge DOF} \\
\end{align*}
\]

\[
\begin{align*}
\tilde{\nabla}_j \tilde{Q}^{ij} = \tilde{Q}_i &= 0 \\
\tilde{Q}^{ij} : & \quad \text{2 dynamical DOF} \\
\end{align*}
\]

\[
\begin{align*}
(LX)^{ij} & \equiv \tilde{\nabla}^i X^j + \tilde{\nabla}^j X^i - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\nabla}_k X^k \\
X^i : & \quad \text{3 constrained DOF} \\
\end{align*}
\]

\[
K^{ij} = \psi^{-10} \left[ (LX)^{ij} + \tilde{Q}^{ij} \right] + \frac{1}{3} \gamma^{ij} K
\]

\[
\begin{align*}
\tilde{\nabla}_j X^i & : \quad \text{3 constrained DOF} \\
K : & \quad \text{1 temporal gauge DOF} \\
\end{align*}
\]
Specifying Initial Data

Freely specified degrees of freedom

\[ \tilde{\gamma}_{ij} \leftarrow \begin{cases} (2) \text{ initial dynamical ("wave") content} \\ (3) \text{ initial spatial gauge choices} \end{cases} \]

\[ \tilde{Q}^{ij} \leftarrow (2) \text{ initial dynamical ("wave") content} \]

\[ K \leftarrow (1) \text{ initial temporal gauge choice} \]

Constrained degrees of freedom

\[ \psi \leftarrow \begin{cases} (1) \text{ Hamiltonian constraint} \\ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \end{cases} \]

\[ X^i \leftarrow \begin{cases} (3) \text{ Momentum constraint} \\ \tilde{\Delta}_L X^i = \frac{2}{3} \psi^6 \tilde{\nabla}^i K + 8\pi \psi^{10} j^i \end{cases} \]

Boundary conditions

The constraints form a set of 4 coupled nonlinear PDEs for \((\psi, X^i)\) that require the specification of boundary conditions at spatial infinity and any interior boundaries.
“Traditional” Black-Hole Data

Conformal flatness and maximal slicing

\[
\begin{align*}
\tilde{\gamma}_{ij} &= f_{ij} \text{ (flat)} \\
\tilde{Q}^{ij} &= 0 \\
K &= 0
\end{align*}
\]

\[
\implies \begin{cases}
\tilde{\Delta} X^i = 0 \\
\tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0
\end{cases}
\]

Bowen-York solution[3]
Analytic solutions for \( \tilde{A}^{ij} \)

Three general solution schemes

Conformal Imaging-[7]
Inversion symmetry inner-BC

Apparent Horizon BC-[12]
Apparent horizon inner-BC

Puncture Method-[4]
No inner-BC: singular behavior factored out

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for \( \tilde{\gamma}_{ij} \) and Bowen-York \( \tilde{A}^{ij} \).
“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

How do we construct improved BH initial data?

We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and $\tilde{Q}^{ij}$]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and $K$]
- boundary conditions on the constrained degrees of freedom [in $\psi$ and $X^i$]

so as to conform to the desired physical content of the initial data.

- For black holes in quasi-circular orbits, we can use the principle of quasi-equilibrium to guide our choices.
- Quasi-equilibrium is a dynamical concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.
Conformal Thin-Sandwich Decomposition[14]

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \]

\[ K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[ (\tilde{L} \beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3} \gamma^{ij} K \begin{cases} \tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij} \quad (\tilde{u}_i^i = 0) \\ \tilde{\alpha} \equiv \psi^{-6} \alpha \end{cases} \]

Hamiltonian Const. \[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \]

Momentum Const. \[ \tilde{\Delta}_L \beta^i - (\tilde{L} \beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} = \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K + \tilde{\alpha} \tilde{\nabla}_j \left( \frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16\pi \tilde{\alpha} \psi^{10} j^i \]

\[ \tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{L} \beta)^{ij} - \tilde{u}^{ij} \right] \]

Constrained vars : \( \psi \) and \( \beta^i \)
Freely specified : \( \tilde{\gamma}_{ij}, \tilde{u}^{ij}, K, \) and \( \tilde{\alpha} \)

\( \tilde{u}^{ij} \) and \( \beta^i \) have a simple physical interpretation, unlike \( \tilde{Q}^{ij} \) and \( X^i \).

Quasi-equilibrium \( \Rightarrow \begin{cases} \tilde{u}^{ij} = 0 \\ \partial_t K = 0 \ (\text{Const. on } \alpha) \end{cases} \)

Constr. Tr(\( K \)) eqn. \[ \tilde{\nabla}^2 (\alpha \psi) - \alpha \left[ \frac{1}{8} \psi \tilde{R} + \frac{5}{12} \psi^5 K^2 + \frac{7}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} \right. \\
+ 2\pi \psi^5 K (\rho + 2S) \] \[ = \psi^5 \beta^i \tilde{\nabla}_i K \]
Equations of Quasi-Equilibrium

\[
\begin{align*}
\text{Ham. & Mom. const. eqns. from Conf. TS} \\
+ \text{Const. } \text{Tr}(K) \text{ eqn.}
\end{align*}
\Rightarrow \text{Eqns. of Quasi-Equilibrium}
\]

With \(\tilde{\gamma}_{ij} = f_{ij}, \tilde{u}^{ij} = 0,\) and \(K = 0,\) these equations have been widely used to construct binary neutron star initial data\([1, 11, 2, 13].\)

Binary neutron star initial data require:

• boundary conditions at infinity compatible with asymptotic flatness and corotation.
  \[
  \psi|_{r \to \infty} = 1, \quad \beta^i|_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^i, \quad \alpha|_{r \to \infty} = 1
  \]

• compatible solution of the equations of hydrostatic equilibrium. (\(\Rightarrow \Omega\))

Binary black hole initial data require:

• a means for choosing the angular velocity of the orbit \(\Omega.\)

\* with excision, inner boundary conditions are needed for \(\psi, \beta^i,\) and \(\tilde{\alpha}.\)

Gourgoulhon, Grandclément, & Bonazzola\([9, 10]:\) Black-hole binaries with \(\tilde{\gamma}_{ij} = f_{ij}, \tilde{u}^{ij} = 0, K = 0,\) “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require constraint violating regularity condition imposed on inner boundaries!
Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ & $\tilde{K} = 0$ on the inner boundary.

- It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

How do we proceed?

- Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & $\tilde{K}$.

- Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.

\begin{equation}
\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{L}_\beta)^{ij} - \tilde{u}^{ij} \right]
\end{equation}

Approach

- Demand that the excision (inner) boundary be an apparent horizon.

- Demand that the apparent horizon be in quasi-equilibrium.
The Inner Boundary

\[
\begin{align*}
\Sigma_{\mu \nu} & \equiv -\frac{1}{2} h_\mu^{\alpha} h_\nu^{\beta} J_{\kappa \gamma} g_{\alpha \beta} \\
\dot{\Sigma}_{\mu \nu} & \equiv -\frac{1}{2} h_\mu^{\alpha} h_\nu^{\beta} \mathcal{L}_k g_{\alpha \beta}
\end{align*}
\]

Extrinsic curvature of \( S \) embedded in \( \Sigma \)

\[
H_{ij} \equiv -\frac{1}{2} J_i^k J_j^\ell \mathcal{L}_s \gamma_{k \ell}
\]

Projections of \( K_{ij} \) onto \( S \)

\[
\begin{align*}
J_{ij} & \equiv h_i^k h_j^\ell K_{k \ell} \\
J_i & \equiv h_i^k s^\ell K_{k \ell} \\
J & \equiv h^{ij} J_{ij} = h^{ij} K_{ij}
\end{align*}
\]

Expansion of null rays

\[
\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (J + H)
\]

Shear of null rays

\[
\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta
\]

\[
\dot{\theta} \equiv h^{ij} \dot{\Sigma}_{ij} = \frac{1}{\sqrt{2}} (J - H)
\]

\[
\dot{\sigma}_{ij} \equiv \dot{\Sigma}_{ij} - \frac{1}{2} h_{ij} \dot{\theta}
\]
AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $\mathcal{S}$ is a (MOTS):
   \[ \theta = 0 \]

2. The inner boundary $\mathcal{S}$ doesn’t move:
   \[ \mathcal{L}_\zeta \tau = 0 \text{ and } \nabla_i \mathcal{L}_\zeta \tau \equiv h^j_i \nabla_j \mathcal{L}_\zeta \tau = 0 \]
   \[ t^\mu = \alpha n^\mu + \beta^\mu \]
   \[ \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu \]
   \[ \beta_\perp \equiv \beta^i s_i \]

3. The inner boundary $\mathcal{S}$ remains a MOTS[8]:
   \[ \mathcal{L}_\zeta \theta = 0 \text{ and } \mathcal{L}_\zeta \dot{\theta} = 0 \]

4. The horizons are in quasi-equilibrium:
   \[ \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S} \]
Evolution of the Expansions

\[ \mathcal{L}_\zeta \theta = \frac{1}{\sqrt{2}} \left[ \theta (\theta + \frac{1}{2} \dot{\theta} - \frac{1}{\sqrt{2}} K) + \mathcal{E} \right] (\beta_\perp + \alpha) \]
\[ + \frac{1}{\sqrt{2}} \left[ \theta (\frac{1}{2} \theta - \frac{1}{2} \dot{\theta} - \frac{1}{\sqrt{2}} K) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu k^\nu \right] (\beta_\perp - \alpha) \]
\[ + \theta s^i \tilde{\nabla}_i \alpha, \]

\[ \mathcal{L}_\zeta \dot{\theta} = -\frac{1}{\sqrt{2}} \left[ \dot{\theta} (\dot{\theta} + \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K) + \dot{\mathcal{E}} \right] (\beta_\perp - \alpha) \]
\[ - \frac{1}{\sqrt{2}} \left[ \dot{\theta} (\frac{1}{2} \dot{\theta} - \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K) + \dot{\mathcal{D}} + 8\pi T_{\mu\nu} k^\mu k^\nu \right] (\beta_\perp + \alpha) \]
\[ - \dot{\theta} s^i \tilde{\nabla}_i \alpha, \]

\[ \mathcal{D} \equiv h^{ij}(\tilde{\nabla}_i + J_i)(\tilde{\nabla}_j + J_j) - \frac{1}{2} \hat{R} \]
\[ \dot{\mathcal{D}} \equiv h^{ij}(\tilde{\nabla}_i - J_i)(\tilde{\nabla}_j - J_j) - \frac{1}{2} \hat{R} \]
\[ \mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu \]
\[ \dot{\mathcal{E}} \equiv \dot{\sigma}_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} \dot{k}^\mu \dot{k}^\nu \]

Incorporates the constraint and evolution equations of GR, the Gauss–Codazzi–Ricci equations governing the embedding of \( S \) in the spatial hypersurface, and the demand that \( S \) remain at a constant coordinate location. *These equations incorporate no assumption of quasi-equilibrium.*

“Red” terms vanish because we demand \( S \) be a MOTS, remain a MOTS, or because we demand the horizon to be in equilibrium.
AH/Quasi-Equilibrium Boundary Conditions

\[ \theta = 0 \]

\[ 0 = \mathcal{D}(\beta_\perp - \alpha), \]

\[ \dot{s}^i \nabla_i \alpha = -\frac{1}{\sqrt{2}} \left[ \dot{\theta}(\theta - \frac{1}{\sqrt{2}} K) + \dot{\sigma}_{ij} \dot{\sigma}^{ij} \right] (\beta_\perp - \alpha) \]

\[ -\frac{1}{\sqrt{2}} \left[ \dot{\theta}(\frac{1}{2}\dot{\theta} - \frac{1}{\sqrt{2}} K) + \dot{\mathcal{D}} \right] (\beta_\perp + \alpha). \]

\[ \tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4}(\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \]

\[ \beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta^i_\parallel \]

\[ J \tilde{s}^i \tilde{\nabla}_i \alpha = -\psi^2 (J^2 - JK + \tilde{\mathcal{D}}) \alpha \]

\( h_{ij} \equiv \psi^4 \tilde{h}_{ij} \)

\( s^i \equiv \psi^{-2} \tilde{s}^i \)

\[ \beta^i_\parallel s_i = 0 \]

\[ \tilde{\mathcal{D}} \equiv \psi^{-4}[\tilde{h}^{ij} (\tilde{\nabla}_i - J_i)(\tilde{\nabla}_j - J_j) - \frac{1}{2} \tilde{R} + 2 \tilde{\nabla}^2 \ln \psi] \]

\[ [\tilde{\nabla} \& \tilde{R} \text{ are compatible with } \tilde{h}_{ij}] \]

The conditions of quasi-equilibrium yield boundary conditions for 3 of the 5 constrained variables (\( \psi, \alpha, \beta_\perp \)). The remaining two conditions are contained in the definition of \( \beta^i_\parallel \). This freedom is necessary to prescribe the spin of the black hole.
Defining the Spin of the Black Hole

The spin parameters $\beta_i^\parallel$ can be defined by demanding that the MOTS be a *Killing horizon*. The time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^\mu \propto (n^\mu + s^\mu) \implies k^\mu = \left[1, \alpha s^i - \beta^i\right]$$

This vector $k^\mu$ is null for any choice of $\alpha$ & $\beta^i$.

In the frame where a black hole is not spinning, the null time vector has components $t^\mu = [1, \vec{0}]$.

**Corotating Holes**

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta^i_\parallel = 0$$

**Irrotational Holes**

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi}\right)^i \implies \beta^i_\parallel = \Omega \left(\frac{\partial}{\partial \phi}\right)^i$$
Summary of QE Formalism

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0
\]

\[
\Delta_{\mathbb{L}_\beta}^i - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K
\]

\[
\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[ \frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} \tilde{A}_{ij} \tilde{A}^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0
\]

\[
\tilde{s}^k \tilde{\nabla}_k \ln \psi \big|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \big|_S \quad \theta = 0
\]

\[
\beta^i \big|_S = \begin{cases} 
\alpha \psi^{-2} \tilde{s}^i \big|_S \\
\alpha \psi^{-2} \tilde{s}^i \big|_S + \Omega \tilde{h}_j^i \left( \frac{\partial}{\partial \phi} \right)^j \bigg|_S
\end{cases}
\]

\[
\text{corotation} \quad \mathcal{L}_\zeta \theta = 0 \quad \sigma_{ij} = 0
\]

\[
\text{irrotation} \quad J \tilde{s}^i \tilde{\nabla}_i \alpha \big|_S = -\psi^2 (J^2 - JK + \tilde{D}) \alpha \big|_S \quad \mathcal{L}_\zeta \hat{\theta} = 0
\]

\[
\psi \big|_{r \to \infty} = 1 \quad \beta^i \big|_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^i \quad \alpha \big|_{r \to \infty} = 1
\]

The only remaining freedom in the system is the choice of the orbital angular velocity, the initial spatial and temporal gauge, and the initial dynamical ("wave") content found in \( \Omega, \tilde{\gamma}_{ij} \) and \( K \).
The Orbital Angular Velocity

- For a given choice of $\tilde{\gamma}_{ij}$ and $K$, we are still left with a family of solutions parameterized by the orbital angular velocity $\Omega$.
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of $\Omega$ to correspond to a system in quasi-equilibrium.

GGB[9, 10] have suggested a way to pick the quasi-equilibrium value of $\Omega$:

$\Omega$ is chosen as the value for which the ADM energy $E_{ADM}$ equals the Komar mass $M_K$.

\[
M_K = \frac{1}{4\pi} \int_{\infty} \tilde{\gamma}_{ij} (\tilde{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j
\]

Acceptable definition of the mass only for stationary spacetimes.

\[
E_{ADM} = \frac{1}{16\pi} \int_{\infty} \gamma^{ij} \tilde{\nabla}_k (\mathcal{G}^k_i - \delta^k_i \mathcal{G}) d^2 S_j
\]

Acceptable definition of the mass for arbitrary spacetimes.

$\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$
Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests: Implementation and results due to H. Pfeiffer

- $\tilde{\gamma}_{ij}$ and $K$ from Kerr-Schild:
  - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.
  - Replacing the BC on either $\alpha$ or $\beta_\perp$ with the proper Dirichlet data yields good convergence.
  - Replacing the BC on either $\alpha$ or $\beta_\perp$ with the wrong Dirichlet data yields good convergence.
  - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
    * obeys the full AH/QE BCs
    * has $\partial_t \psi = 0$
    *(if the outer boundary is at $\infty$)*

- $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 1/r^2$ or 0
  - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
    * obeys the full AH/QE BCs
    * has $\partial_t \psi = 0$
    *(if the outer boundary is at $\infty$)*
Fixing the Length Scale

These results suggest that, at least in spherical symmetry, the AH/QE BC’s yield a one-parameter family of solutions. How do we fix a unique solution?

- Modify the BC on $\psi$ so that it also fixes the average value of $\psi$:
  - AH/QE BCs + $\bar{\psi}$ converges if the initial guess is “good”.
  - With a poor initial guess, the solution gets caught in a local minimum.

- Modify the BC on $\alpha$ so that it also fixes the average value of $\alpha$:
  - AH/QE BCs + $\bar{\alpha}$ converges if the initial guess is “good”.
  - With a poor initial guess, the solution gets caught in a local minimum.

Convergence problems are fixed for general, non-spherical cases if $\bar{\psi}$ or $\bar{\alpha}$ are fixed and one term in QE BC on $\alpha$ is changed:

- $\tilde{h}^{ij} \tilde{\nabla}_i J_j \to 0$
- $\tilde{h}^{ij} \tilde{\nabla}_i J_j \to -\tilde{h}^{ij} \tilde{\nabla}_i J_j$

We are still trying to determine if this is a bug in the code or a fundamental problem
References


