Initial Data for Black-Hole Binaries

Gregory B. Cook
Wake Forest University

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Abstract
We will examine the current state of our efforts to generate astrophysically realistic initial data for black-hole binaries.

Collaborators: Harald Pfeiffer (Caltech) & Saul Teukolsky (Cornell)
Motivation

- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, . . .
- Available computed waveforms should increase chance of detecting collision events.

Quasi-Equilibrium Binary Data

- General Relativity doesn’t permit true equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (quasi-circular) then there is a corotating reference frame in which the binary appears to be at rest.

★ Quasi-equilibrium gives us a physical condition to guide us in fixing boundary conditions and data that is not otherwise constrained.
The 3 + 1 Decomposition

Lapse: \( \alpha \)  
Spatial metric: \( \gamma_{ij} \)  
Shift vector: \( \beta^i \)  
Extrinsic Curvature: \( K_{ij} \)  
Time vector: \( t^\mu = \alpha n^\mu + \beta^\mu \)

\[ \text{d}s^2 = -\alpha^2 \text{d}t^2 + \gamma_{ij}(\text{d}x^i + \beta^i \text{d}t)(\text{d}x^j + \beta^j \text{d}t) \]
\[ \gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \]
\[ K_{\mu\nu} = -\frac{1}{2} \gamma^\alpha_{\mu} \gamma^\beta_\nu \mathcal{L}_n g_{\alpha\beta} \]

Constraint equations

\[ \bar{R} + K^2 - K_{ij} K^{ij} = 16\pi\rho \]
\[ \bar{\nabla}_j \left( K^{ij} - \gamma^{ij} K \right) = 8\pi j^i \]

\[ S_{\mu\nu} \equiv \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta} \]
\[ j_{\mu} \equiv -\gamma^\nu_\mu n^\alpha T_{\nu\alpha} \]
\[ \rho \equiv n^{\mu} n^{\nu} T_{\mu\nu} \]
\[ T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu} j_{\nu)} + n_\mu n_\nu \rho \]

Evolution equations

\[ \partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i \]
\[ \partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[ \bar{R}_{ij} - 2 K_{i\ell} K^{\ell j} + K K_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij} (S - \rho) \right] + \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell} \bar{\nabla}_j \beta^\ell + K_{j\ell} \bar{\nabla}_i \beta^\ell \]
Conformal Thin-Sandwich Decomposition

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \]
\[ K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[ (\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K \]

Hamiltonian Const.
\[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \]

Momentum Const.
\[ \tilde{\nabla}_j (\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} = \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K + \tilde{\alpha} \tilde{\nabla}_j \left( \frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16\pi \tilde{\alpha} \psi^{10} j^i \]

Constrained vars: \( \psi, \beta^i, \) and \( \tilde{\alpha} \equiv \psi^{-6} \alpha \)

Freely specified:
\[ K \quad \text{and} \quad \partial_t K \]

Constrained vars:
\[ \tilde{\gamma}_{ij}, \tilde{u}^{ij} \equiv -\partial_t \tilde{\gamma}^{ij} \]

Quasi-equilibrium \( \Rightarrow \)
\[
\begin{cases}
\partial_t \tilde{\gamma}^{ij} = 0 \\
\partial_t K = 0
\end{cases}
\]
Equations of Quasi-Equilibrium

\[
\begin{align*}
\text{Ham. & Mom. const. eqns., & Const Tr}(K) & \quad \text{eqn. from Conf. TS} \\
+ \tilde{u}^{ij} = \partial_t K = 0 
\end{align*}
\]

\[\Rightarrow \text{Eqns. of Quasi-Equilibrium}\]

With \(\tilde{\gamma}_{ij} = f_{ij}\) and \(K = 0\), these equations have been widely used to construct binary neutron star initial data\([2, 9, 3, 10]\).

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.
  \[
  \psi|_{r \to \infty} = 1 \quad \beta^i|_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \to \infty} = 1
  \]

- compatible solution of the equations of hydrostatic equilibrium. \((\Rightarrow \Omega)\)
Equations of Quasi-Equilibrium

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  \]

- compatible solution of the equations of hydrostatic equilibrium. \( \Rightarrow \Omega \)

Binary black hole initial data require:

- a means for choosing the angular velocity of the orbit \( \Omega \).

\* with excision, inner boundary conditions are needed for \( \psi, \beta^i \), and \( \tilde{\alpha} \).

Gourgoulhon, Grandclément, & Bonazzola\([7, 8]\): Black-hole binaries with \( \tilde{\gamma}_{ij} = f_{ij} \) & \( K = 0 \), “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require constraint violating regularity condition imposed on inner boundaries!
The Inner Boundary

$\Sigma$

Extrinsic curvature of $S$ embedded in $\Sigma$

$H_{ij} \equiv \frac{1}{2} h^k_i h^\ell_j \mathcal{L}_s \gamma_{k\ell}$

Projections of $K_{ij}$ onto $S$

$J_{ij} \equiv h^k_i h^\ell_j K_{k\ell}$

$J_i \equiv h^k_i s^\ell K_{k\ell}$

$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$

Expansion of null rays

$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (H - J)$

Shear of null rays

$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$

$\dot{\sigma}_{ij} \equiv \dot{\Sigma}_{ij} - \frac{1}{2} h_{ij} \dot{\theta}$

$\dot{\theta} \equiv h^{ij} \dot{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H + J)$

$s_i \equiv \frac{\nabla_i \tau}{|\nabla \tau|}$

$h_{ij} \equiv \gamma_{ij} - s_is_j$

$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$

$\dot{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$

Extrinsic curvature of $\Sigma$ embedded in spacetime

$\Sigma_{\mu\nu} \equiv \frac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_k g_{\alpha\beta}$

$\dot{\Sigma}_{\mu\nu} \equiv \frac{1}{2} h^\alpha_\mu h^\beta_\nu \dot{\mathcal{L}}_k g_{\alpha\beta}$

Extrinsic curvature of $S$ embedded in $\Sigma$

$\Sigma_{ij} = \frac{1}{\sqrt{2}} (H_{ij} - J_{ij})$

$\dot{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H_{ij} + J_{ij})$
AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $S$ is a (MOTS):  
   \[ \theta = 0 \]

2. The inner boundary $S$ remains a MOTS:  
   \[ \mathcal{L}_\zeta \theta = 0 \]
   \[ t^\mu = \alpha n^\mu + \beta^\mu \]
   \[ \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu \]
   \[ \beta_\perp \equiv \beta^i s_i \]
   $\zeta^\mu$ is null on the AH and the chosen form is a gauge choice.

3. The horizons are in quasi-equilibrium:  
   \[ \sigma_{ij} = 0 \] and no matter is on $S$
AH/Quasi-Equilibrium Boundary Conditions

\[ \theta = \frac{\psi^{-2}}{\sqrt{2}} \left[ \tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4 \tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right] \]

\[ \mathcal{L}_\zeta \theta = -\frac{1}{\sqrt{2}} \left[ \theta \left( \theta + \frac{1}{2} \dot{\theta} + \frac{1}{\sqrt{2}} K \right) + \mathcal{E} \right] (\beta_\perp + \alpha) \]

\[ -\frac{1}{\sqrt{2}} \left[ \theta \left( \frac{1}{2} \theta - \frac{1}{2} \dot{\theta} + \frac{1}{\sqrt{2}} K \right) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu k^\nu \right] (\beta_\perp - \alpha) \]

\[ + \theta s^i \tilde{\nabla}_i \alpha \]

\[ \mathcal{D} \equiv \tilde{h}^{ij} (D_i + J_i)(D_j + J_j) - \frac{1}{2} R \]

\[ \mathcal{E} \equiv \sigma_{ij} \dot{\sigma}^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu \]

\[ \sigma_{ij} = \frac{1}{\sqrt{2}} \left( H_{ij} - \frac{1}{2} h_{ij} H \right) \left( 1 - \frac{\beta_\perp}{\alpha} \right) \]

\[ -\frac{1}{\sqrt{2}} \frac{\psi^4}{\alpha} \left\{ \tilde{D}_{(i\beta\|j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_{k\|i} \beta^k - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{j\ell} \tilde{u}^{k\ell} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{k\ell} \tilde{u}^{k\ell}] \right\} \]
AH/Quasi-Equilibrium Boundary Conditions

$$\theta = \frac{\psi^{-2}}{\sqrt{2}} \left[ \tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4 \tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right]$$

$$\mathcal{L}_\zeta \theta = -\frac{1}{\sqrt{2}} \left[ \theta (\theta + \frac{1}{2} \dot{\theta} + \frac{1}{\sqrt{2}} K) + \mathcal{E} \right] (\beta_\perp + \alpha)$$

$$-\frac{1}{\sqrt{2}} \left[ \theta (\frac{1}{2} \dot{\theta} - \frac{1}{2} K) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu \dot{k}^\nu \right] (\beta_\perp - \alpha)$$

$$+ \theta s^i \tilde{\nabla}_i \alpha$$

$$\mathcal{D} \equiv \tilde{h}^{ij} (D_i + J_i) (D_j + J_j) - \frac{1}{2} R$$

$$\mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu$$

$$\sigma_{ij} = \frac{1}{\sqrt{2}} \left( H_{ij} - \frac{1}{2} h_{ij} H \right) \left( 1 - \frac{\beta_\perp}{\alpha} \right)$$

$$- \frac{1}{\sqrt{2}} \psi^4 \left\{ \tilde{D}_{(i \beta_\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_\parallel^k - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{j\ell} \tilde{u}^{k\ell} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{k\ell} \tilde{u}^{k\ell}] \right\}$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)$$

$$\beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_\parallel^i$$

$$0 = \tilde{D}_{(i \beta_\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_\parallel^k$$
AH/Quasi-Equilibrium Boundary Conditions

\[ \theta = \frac{\psi^{-2}}{\sqrt{2}} \left[ \tilde{h}_{ij} \tilde{\nabla}_i \tilde{s}_j + 4 \tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right] \]

\[ \mathcal{L}_\zeta \theta = - \frac{1}{\sqrt{2}} \left[ \theta (\theta + \frac{1}{2} \dot{\theta} + \frac{1}{\sqrt{2}} K) + \mathcal{E} \right] (\beta_\perp + \alpha) \]
\[ - \frac{1}{\sqrt{2}} \left[ \theta \left( \frac{1}{2} \theta - \frac{1}{2} \dot{\theta} + \frac{1}{\sqrt{2}} K \right) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu \dot{\tilde{k}}^\nu \right] (\beta_\perp - \alpha) \]

\[ + \theta s^i \tilde{\nabla}_i \alpha \]

\[ \mathcal{D} \equiv \tilde{h}_{ij} (D_i + J_i) (D_j + J_j) - \frac{1}{2} R \]

\[ \mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu \]

\[ \sigma_{ij} = \frac{1}{\sqrt{2}} \left( H_{ij} - \frac{1}{2} h_{ij} H \right) \left( 1 - \frac{\beta_\perp}{\alpha} \right) \]
\[ - \frac{1}{\sqrt{2}} \frac{\psi^4}{\alpha} \left\{ \tilde{D}_i \beta_{\parallel j} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_\parallel^k - \frac{1}{2} \left[ \tilde{h}_{ik} \tilde{h}_{j\ell} \tilde{u}^{k\ell} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{k\ell} \tilde{u}^{k\ell} \right] \right\} \]

\[ \tilde{s}^k \tilde{\nabla}_k \ln \psi = - \frac{1}{4} (\tilde{h}_{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \]

\[ \beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_{\parallel}^i \]

\[ 0 = \tilde{D}_i \beta_{\parallel j} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_\parallel^k \]

\[ \partial_t \ln \psi = \left[ \tilde{D}_k \beta_\parallel^k + 4 \beta_{\parallel}^k \tilde{D}_k \ln \psi \right. \]
\[ - \frac{1}{2} \tilde{h}_{k\ell} \tilde{u}^{k\ell} \]
\[ \left. - \sqrt{2} \theta + (\beta_\perp - \alpha) H \right] \]
Defining the Spin of the Black Hole

The spin parameters $\beta^i$ can be defined by demanding that the time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^\mu \propto (n^\mu + s^\mu) \implies k^\mu = \left[1, \alpha s^i - \beta^i\right]$$

This vector $k^\mu$ is null for any choice of $\alpha$ & $\beta^i$.

In the frame where a black hole is not spinning, the null time vector has components $t^\mu = [1, \vec{0}]$.

### Corotating Holes
Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta^i = 0$$

### Irrotational Holes
Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$$

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, -\Omega(\partial/\partial \phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi}\right)^i \implies \beta_{\parallel}^i = \Omega \xi^i$$

$$\xi^i \approx \left(\frac{\partial}{\partial \phi}\right)^i \quad \& \quad \tilde{D}_{(i \xi_j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \xi^k = 0$$
The Lapse BC & QE

So far, nothing has fixed a boundary condition on the lapse $\alpha$. One possibility\cite{6} is to recall that $\theta\dot{\theta}$ is a Lorentz invariant and so to consider $\mathcal{L}_\zeta \dot{\theta} = 0$ as a quasi-equilibrium condition.

\[
\mathcal{L}_\zeta \dot{\theta} = 0 \quad \Rightarrow \quad J s^i \tilde{\nabla}_i \alpha = -\psi^2 (J^2 - J K + \tilde{D}) \alpha
\]

\[
\tilde{D} \equiv \psi^{-4} [\tilde{h}^{ij} (\tilde{D}_i - J_i)(\tilde{D}_j - J_j) - \frac{1}{2} \tilde{R} + 2 \tilde{D}^2 \ln \psi]
\]
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$$\mathcal{L}_\zeta \dot{\theta} = 0 \quad \implies \quad J \tilde{s}^i \tilde{\nabla}_i \alpha = -\psi^2 (J^2 - JK + \tilde{D}) \alpha$$

$$\tilde{D} \equiv \psi^{-4} [\tilde{h}^{ij} (\tilde{D}_i - J_i)(\tilde{D}_j - J_j) - \frac{1}{2} \tilde{R} + 2 \tilde{D}^2 \ln \psi]$$

This condition is satisfied for stationary solutions, but seems to be degenerate with the other QE boundary conditions. To see this, note that the stationary maximal slicings of Schwarzschild form a 1-parameter family:

$$ds^2 = \frac{dR^2}{1 - \frac{2M}{R} + \frac{C^2}{R^4}} + R^2 d^2\Omega$$

$$\alpha = \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}$$

$$\beta^R = \frac{C}{R^2} \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}$$

$$K^i_j = \frac{C}{R^3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\alpha|_S = \frac{C}{4M^2}$$
The Orbital Angular Velocity

- For a given choice of the Lapse BC, $\tilde{\gamma}_{ij}$ and $K$, we are still left with a family of solutions parameterized by the orbital angular velocity $\Omega$.
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of $\Omega$ to correspond to a system in quasi-equilibrium.

GGB[7, 8] have suggested a way to pick the quasi-equilibrium value of $\Omega$:

\[ \Omega \text{ is chosen as the value for which the ADM energy } E_{\text{ADM}} \text{ equals the Komar mass } M_K. \]

\[
M_K = \frac{1}{4\pi} \int_{\infty} \gamma^{ij} (\bar{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j \quad \text{Acceptable definition of the mass only for stationary spacetimes.}
\]

\[
E_{\text{ADM}} = \frac{1}{16\pi} \int_{\infty} \gamma^{ij} \bar{\nabla}_k (G^k_i - \delta^k_i G) d^2 S_j \quad \text{Acceptable definition of the mass for arbitrary spacetimes.} \\
G_{ij} \equiv \gamma_{ij} - f_{ij}
\]
Summary of QE Formalism

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{L}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \\
\tilde{\nabla}_j (\tilde{L}\beta)^{ij} - (\tilde{L}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K \\
\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi)\left[ \frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} A_{ij} A^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0
\]

\[
\tilde{s}^k \tilde{\nabla}_k \ln \psi|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \theta = 0
\]

\[
\beta^i|_S = \begin{cases} 
\alpha \psi^{-2} \tilde{s}^i|_S & \text{corotation} \\
\alpha \psi^{-2} \tilde{s}^i|_S + \Omega \xi^i|_S & \text{irrotation}
\end{cases}
\]

\[
\alpha|_S = \text{unspecified by QE}
\]

| \psi|_{r \to \infty} = 1 \\
|\beta^i|_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^i \\
|\alpha|_{r \to \infty} = 1
|}

The only remaining freedom in the system is the choice of the lapse boundary condition, the initial spatial and temporal gauge, and the initial dynamical ("wave") content found in \( \alpha|_S, \tilde{\gamma}_{ij} \) and \( K \).
Results

Corotation
\[ \tilde{\gamma}_{ij} = f_{ij} \] : Maximal Slicing:
- \[ \frac{\partial (\alpha \psi)}{\partial r} = 0 \]
- \[ \alpha \psi = \frac{1}{2} \]
- \[ \frac{\partial (\alpha \psi)}{\partial r} = \frac{\alpha \psi}{2r} \]

\[ \tilde{\gamma}_{ij} = f_{ij} \] : Eddington-Finkelstein Slicing:
- \[ \frac{\partial (\alpha \psi)}{\partial r} = 0 \]
- \[ \alpha \psi = \frac{1}{2} \]
- \[ \frac{\partial (\alpha \psi)}{\partial r} = \frac{\alpha \psi}{2r} \]

Irrotation
\[ \tilde{\gamma}_{ij} = f_{ij} \] : Maximal Slicing:
- \[ \frac{\partial (\alpha \psi)}{\partial r} = 0 \]
- \[ \alpha \psi = \frac{1}{2} \]
- \[ \frac{\partial (\alpha \psi)}{\partial r} = \frac{\alpha \psi}{2r} \]

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- \[ \frac{\partial (\alpha \psi)}{\partial r} = \frac{\alpha \psi}{2r} \]

Compared with
- Effective-One-Body PN[5]
- Inversion-Symmetric HKV[8]

Compared with
- Effective-One-Body PN[5]
- Conformal Imaging[4]
- Puncture Method[1]
Corotating; Maximal Slicce; QE-BC; $E_b/\mu$ vs $J/\mu m$
Corotating; Maximal Slice; Comparison; $E_b/\mu$ vs $J/\mu m$
Irrotational; Maximal Slice; QE-BC; $E_b/\mu$ vs $J/\mu m$

\begin{align*}
\text{IR: MS - } d(\alpha \psi)/dr &= 0 \\
\text{IR: MS - } \alpha \psi &= 1/2 \\
\text{IR: MS - } d(\alpha \psi)/dr &= (\alpha \psi)/2r
\end{align*}
Irrotational; Maximal Slicce; Comparison; $E_b/\mu$ vs $J/\mu m$
Maximal Slice; Comparison of ISCO; \( \frac{E_b}{M_{irr}} \) vs \( \Omega M_{irr} \)
Maximal Slice; Comparison of ISCO; $E_b/M_{irr}$ vs $J/M_{irr}^2$
Open Questions

• Is the physics of the corotating and irrotational models correct?
  – Do the corotating black holes have the correct angular momentum?
  – Is the angular momentum of the irrotational holes nearly zero?

• How do we make a *physically motivated* choice for $\tilde{\gamma}_{ij}$?
References


