Computing Initial Data I: Formalisms and History

Gregory B. Cook

Wake Forest University

June. 27, 2002

Abstract

In the first of two talks on the computation of initial data, we will look at some of the formalisms used for posing the constraint equations of general relativity as a boundary-value problem. This process requires making well-motivated choices for which of the initial-data quantities are constrained and which can be freely specified. After looking at the general formalisms used to construct initial data, we will review the approaches that have been used to date in constructing black-hole and neutron-star initial data. Finally we will look at some of the current issues, from a physicist perspective, that are at the forefront of initial-data research.

A related review article is online at Living Review in Relativity[23]
The $3 + 1$ Decomposition

Lapse: $\alpha$  
Spatial metric: $\gamma_{ij}$

Shift vector: $\beta^i$  
Extrinsic Curvature: $K_{ij}$

Time vector: $t^\mu = \alpha n^\mu + \beta^\mu$

$$\begin{align*}
ds^2 &= -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt) \\
\gamma_{\mu\nu} &= g_{\mu\nu} + n_\mu n_\nu \\
K_{\mu\nu} &= -\frac{1}{2} \gamma_{\mu}^{\alpha} \gamma_{\nu}^{\beta} \mathcal{L}_n g_{\alpha\beta}
\end{align*}$$
**The 3 + 1 Decomposition**

Lapse: $\alpha$  
Spatial metric: $\gamma_{ij}$  
Shift vector: $\beta^i$  
Extrinsic Curvature: $K_{ij}$  
Time vector: $t^\mu = \alpha n^\mu + \beta^\mu$

\[ \text{ds}^2 = -\alpha^2 \text{d}t^2 + \gamma_{ij} (\text{d}x^i + \beta^i \text{d}t)(\text{d}x^j + \beta^j \text{d}t) \]

\[
\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad K_{\mu\nu} = -\frac{1}{2} \gamma^\alpha_\mu \gamma^\beta_\nu \mathcal{L}_n g_{\alpha\beta}
\]

**Constraint equations**

\[
\bar{R} + K^2 - K_{ij} K^{ij} = 16\pi \rho
\]

\[
\bar{\nabla}_j \left( K^{ij} - \gamma^{ij} K \right) = 8\pi j^i
\]

\[
S_{\mu\nu} \equiv \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta}
\]

\[
j_\mu \equiv -\gamma^\nu_\mu n^\alpha T_{\nu\alpha}
\]

\[
\rho \equiv n^\mu n^\nu T_{\mu\nu}
\]

\[
T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu} j_{\nu)} + n_\mu n_\nu \rho
\]

**Evolution equations**

\[
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i
\]

\[
\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[ \bar{R}_{ij} - 2K_{i\ell} K_j^\ell + KK_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij} (S - \rho) \right]
\]

\[
+ \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell} \bar{\nabla}_j \beta^\ell + K_{j\ell} \bar{\nabla}_i \beta^\ell
\]
Degrees of Freedom

**Kinematical variables**
- Lapse $\alpha$ : 1 degree of freedom
- Shift $\beta^i$ : 3 degrees of freedom

**Initial-data variables**
- Metric $\gamma_{ij}$ : 6 degrees of freedom
- Extrinsic curv. $K_{ij}$ : 6 degrees of freedom
Degrees of Freedom

**Kinematical variables**
- Lapse $\alpha$ : 1 degree of freedom
- Shift $\beta^i$ : 3 degrees of freedom

**Initial-data variables**
- Metric $\gamma_{ij}$ : 6 degrees of freedom
- Extrinsic curv. $K_{ij}$ : 6 degrees of freedom

**Simple counting arguments**

Between $\gamma_{ij}$ and $K_{ij}$, there are 12 degrees of freedom at each point in space.
Degrees of Freedom

**Kinematical variables**
- Lapse $\alpha$: 1 degree of freedom
- Shift $\beta^i$: 3 degrees of freedom

**Initial-data variables**
- Metric $\gamma_{ij}$: 6 degrees of freedom
- Extrinsic curv. $K_{ij}$: 6 degrees of freedom

**Simple counting arguments**
Between $\gamma_{ij}$ and $K_{ij}$, there are 12 degrees of freedom at each point in space.

**Dynamical**
Gravitational waves have two polarization states. So, there are $2 + 2 = 4$ dynamical degrees of freedom.
*(1st order form)*
Degrees of Freedom

**Kinematical variables**
- Lapse $\alpha$ : 1 degree of freedom
- Shift $\beta^i$ : 3 degrees of freedom

**Initial-data variables**
- Metric $\gamma_{ij}$ : 6 degrees of freedom
- Extrinsic curv. $K_{ij}$ : 6 degrees of freedom

**Simple counting arguments**

Between $\gamma_{ij}$ and $K_{ij}$, there are 12 degrees of freedom at each point in space.

**Dynamical**
Gravitational waves have two polarization states.
So, there are $2 + 2 = 4$ dynamical degrees of freedom.
(1st order form)

**Gauge**
In a 4-D spacetime, there are 4 initial coordinate gauge choices. So there are $3 + 1 = 4$ gauge degrees of freedom.
Degrees of Freedom

**Kinematical variables**
- Lapse $\alpha$ : 1 degree of freedom
- Shift $\beta^i$ : 3 degrees of freedom

**Initial-data variables**
- Metric $\gamma_{ij}$ : 6 degrees of freedom
- Extrinsic curv. $K_{ij}$ : 6 degrees of freedom

**Simple counting arguments**

Between $\gamma_{ij}$ and $K_{ij}$, there are 12 degrees of freedom at each point in space.

**Dynamical**
Gravitational waves have two polarization states. So, there are $2 + 2 = 4$ *dynamical* degrees of freedom.

*(1st order form)*

**Gauge**
In a 4-D spacetime, there are 4 initial coordinate gauge choices. So there are $3 + 1 = 4$ *gauge* degrees of freedom.

**Constrained**
We have 4 constraint equations and there are $12 - 4 - 4 = 4$ remaining *constrained* degrees of freedom.
Conformal Transverse-Traceless Decomposition

York–Lichnerowicz conformal decomposition of the 3-metric\cite{46, 72, 54}:

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \]

\[
\begin{align*}
\psi & : \quad 1 \text{ constrained DOF} \\
\tilde{\gamma}_{ij} & : \quad 3 \text{ spatial gauge DOF} \\
\tilde{\gamma}_{ij} & : \quad 2 \text{ dynamical DOF}
\end{align*}
\]
Conformal Transverse-Traceless Decomposition

York–Lichnerowicz conformal decomposition of the 3-metric[46, 72, 54]:

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \begin{cases} 
\psi : & 1 \text{ constrained DOF} \\
\tilde{\gamma}_{ij} : & 3 \text{ spatial gauge DOF} \\
X : & 2 \text{ dynamical DOF}
\end{cases}
\]

Conformal TT decomposition of extrinsic curvature[73, 74]:

\[
K^{ij} = \psi^{-10} \left[ (\tilde{\mathcal{L}}X)^{ij} + \tilde{Q}^{ij} \right] + \frac{1}{3} \gamma^{ij} K
\begin{cases} 
\tilde{Q}^{ij} : & \tilde{\nabla}^j \tilde{Q}^{ij} = \tilde{Q}^i_i = 0 \\
X^i : & (\tilde{\mathcal{L}}X)^{ij} \equiv \tilde{\nabla}^i X^j + \tilde{\nabla}^j X^i - \frac{2}{3} \tilde{\gamma}^{ij} \tilde{\nabla}^k X^k \\
K : & 1 \text{ temporal gauge DOF}
\end{cases}
\]
Specifying Initial Data

Freely specified degrees of freedom

\[ \tilde{\gamma}_{ij} \leftarrow \begin{cases} 
& (2) \text{ initial dynamical ("wave") content} \\
& (3) \text{ initial spatial gauge choices} \\
\end{cases} \]

\[ \tilde{Q}^{ij} \leftarrow (2) \text{ initial dynamical ("wave") content} \]

\[ K \leftarrow (1) \text{ initial temporal gauge choice} \]

Freely specified *dynamical gauge freedom*

\[ \beta^i \leftarrow \text{ how spatial coordinates evolve} \]

\[ \alpha \leftarrow \text{ how time coordinate evolves} \]
Specifying Initial Data

Freely specified degrees of freedom

\[ \tilde{\gamma}_{ij} \leftarrow \begin{cases} 
(2) \text{ initial dynamical ("wave") content} \\
(3) \text{ initial spatial gauge choices} 
\end{cases} \]

\[ \tilde{Q}^{ij} \leftarrow (2) \text{ initial dynamical ("wave") content} \]

\[ K \leftarrow (1) \text{ initial temporal gauge choice} \]

Freely specified dynamical gauge freedom

\[ \beta^i \leftarrow \text{how spatial coordinates evolve} \]

\[ \alpha \leftarrow \text{how time coordinate evolves} \]

Constrained degrees of freedom (Conf. TT)

\[ \psi \leftarrow \begin{cases} 
(1) \text{ Hamiltonian constraint} \\
(3) \text{ Momentum constraint} 
\end{cases} \]

\[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \]

\[ \tilde{A}^{ij} \equiv (\tilde{L} X)^{ij} + \tilde{Q}^{ij} \]

\[ \tilde{\Delta}_L X^i \equiv \tilde{\nabla}_j (\tilde{L} X)^{ij} \]

\[ \tilde{\Delta}_L X^i = \frac{2}{3} \psi^6 \tilde{\nabla}^i K + 8\pi \psi^{10} j^i \]
Specifying Initial Data

Freely specified degrees of freedom

\[ \tilde{\gamma}_{ij} \leftarrow \begin{cases} \text{(2) initial dynamical ("wave") content} \\ \text{(3) initial spatial gauge choices} \end{cases} \]

\[ \tilde{Q}^{ij} \leftarrow \text{(2) initial dynamical ("wave") content} \]

\[ K \leftarrow \text{(1) initial temporal gauge choice} \]

Constrained degrees of freedom (Conf. TT)

\[ \psi \leftarrow \begin{cases} \text{(1) Hamiltonian constraint} \\ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \end{cases} \]

\[ X^i \leftarrow \begin{cases} \text{(3) Momentum constraint} \\ \tilde{\Delta}_L X^i = \frac{2}{3} \psi^6 \tilde{\nabla}^i K + 8\pi \psi^{10} j^i \end{cases} \]

Boundary conditions

The constraints form a set of 4 coupled nonlinear PDEs for \((\psi, X^i)\) that require the specification of boundary conditions at spatial infinity and any interior boundaries.
General Initial-Data Decompositions

Conformal TT\([75, 77]\):

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} : K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K : \tilde{A}^{ij} \equiv (\bar{L} V)^{ij} + \tilde{M}^{ij}
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \bar{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho
\]

\[
\tilde{\Delta} L V^i - \frac{2}{3} \psi^6 \tilde{\nabla}^i K = -\tilde{\nabla} j \tilde{M}^{ij} + 8\pi \psi^{10} j^i
\]

Physical TT\([53, 54, 55]\):

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} : K^{ij} = \psi^{-4} \left( \tilde{A}^{ij} + \frac{1}{3} \tilde{\gamma}^{ij} K \right) : \tilde{A}^{ij} \equiv (\bar{L} V)^{ij} + \psi^{-6} \tilde{M}^{ij}
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \bar{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^5 \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho
\]

\[
\tilde{\Delta} L V^i + 6(\bar{L} V)^{ij} \tilde{\nabla} j \ln \psi + \psi^{-6} \tilde{\nabla} j \tilde{M}^{ij} = \frac{2}{3} \tilde{\nabla}^i K + 8\pi \psi^4 j^i
\]

\(\tilde{M}^{ij}\) is symmetric-tracefree, but not divergenceless. The new variable \(V^i\) incorporates the solution of the constraints and the decomposition of \(\tilde{M}^{ij}\) into \(\tilde{Q}^{ij}\).
# General Initial-Data Decompositions

**Conformal TT\([75, 77]\):**

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} : \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K : \quad \tilde{A}^{ij} \equiv (\tilde{L} V)^{ij} + \tilde{M}^{ij}
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2 \pi \psi^5 \rho
\]

\[
\tilde{\Delta}_{L} V^i - \frac{2}{3} \psi^6 \tilde{\nabla}^i K = -\tilde{\nabla}_j \tilde{M}^{ij} + 8 \pi \psi^{10} j^i
\]

**Physical TT\([53, 54, 55]\):**

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} : \quad K^{ij} = \psi^{-4} \left( \tilde{A}^{ij} + \frac{1}{3} \tilde{\gamma}^{ij} K \right) : \quad \tilde{A}^{ij} \equiv (\tilde{L} V)^{ij} + \psi^{-6} \tilde{M}^{ij}
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^5 \tilde{A}_{ij} \tilde{A}^{ij} = -2 \pi \psi^5 \rho
\]

\[
\tilde{\Delta}_{L} V^i + 6 (\tilde{L} V)^{ij} \tilde{\nabla}_j \ln \psi + \psi^{-6} \tilde{\nabla}_j \tilde{M}^{ij} = \frac{2}{3} \tilde{\nabla}^i K + 8 \pi \psi^4 j^i
\]

\(\tilde{M}^{ij}\) is symmetric-tracefree, but not divergenceless. The new variable \(V^i\) incorporates the solution of the constraints and the decomposition of \(\tilde{M}^{ij}\) into \(\tilde{Q}^{ij}\).

**Conformal Thin Sandwich(TS)\([76]\):**

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} : \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K : \quad \tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left( (\tilde{L} \beta)^{ij} + \tilde{u}^{ij} \right)
\]

\[
\tilde{\alpha} \equiv \psi^{-6} \alpha \quad : \quad \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2 \pi \psi^5 \rho
\]

\[
\tilde{\Delta}_{L} \beta^i - (\tilde{L} \beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} - \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K = \tilde{\alpha} \tilde{\nabla}_j \left( \frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16 \pi \tilde{\alpha} \psi^{10} j^i
\]

\(\alpha\) and \(\beta^i\) are the lapse and shift. \(\tilde{u}^{ij}\) is symmetric-tracefree.
### General Initial-Data Decompositions

**Conformal TT[75, 77]:**

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad : \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K \quad : \quad \tilde{A}^{ij} \equiv (\tilde{L}V)^{ij} + \tilde{M}^{ij}
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \\
\tilde{\Delta}_L V^i - \frac{2}{3} \psi^6 \tilde{\nabla}^i K = -\tilde{\nabla}_j \tilde{M}^{ij} + 8\pi \psi^{10} j^i
\]

\[
\tilde{M}^{ij} \text{ is symmetric-tracefree, but not divergenceless. The new variable } V^i \text{ incorporates the solution of the constraints and the decomposition of } \tilde{M}^{ij} \text{ into } \tilde{Q}^{ij}.
\]

**Physical TT[53, 54, 55]:**

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad : \quad K^{ij} = \psi^{-4} \left( \tilde{A}^{ij} + \frac{1}{3} \tilde{\gamma}^{ij} K \right) \quad : \quad \tilde{A}^{ij} \equiv (\tilde{L}V)^{ij} + \psi^{-6} \tilde{M}^{ij}
\]

\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^5 \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \\
\tilde{\Delta}_L V^i + 6(\tilde{L}V)^{ij} \tilde{\nabla}_j \ln \psi + \psi^{-6} \tilde{\nabla}_j \tilde{M}^{ij} = \frac{2}{3} \tilde{\nabla}^i K + 8\pi \psi^4 j^i
\]

\[
\tilde{\alpha} = \frac{1}{2}
\]

\[
\alpha = \frac{1}{2}
\]

**Conformal Thin Sandwich(TS)[76]:**

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad : \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K \quad : \quad \tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left( (\tilde{L}\beta)^{ij} + \tilde{u}^{ij} \right)
\]

\[
\tilde{\alpha} \equiv \psi^{-6} \alpha \quad : \quad \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho
\]

\[
\tilde{\Delta}_L \beta^i - (\tilde{L}\beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} - \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K = \tilde{\alpha} \tilde{\nabla}_j \left( \frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16\pi \tilde{\alpha} \psi^{10} j^i
\]

\[
\alpha \text{ and } \beta^i \text{ are the lapse and shift. } \tilde{u}^{ij} \text{ is symmetric-tracefree.}
\]

– Greg Cook – (WFU Physics)
• The Conformal TS, Conformal TT, & Physical TT decompositions are widely used, but are not unique. In particular, different conformal weightings of the extrinsic curvature are used.

• For stationary situation (i.e. axisymmetric, time independent) like isolated neutron stars, different decompositions are used.

$$ds^2 = -\psi^{-4}dt^2 + \psi^4[A^2(dr^2 + r^2d\theta^2) + B^2r^2\sin^2 \theta(d\phi + \beta^\phi dt)^2]$$

★ The 4 constraints yield 2 independent equations that fix $\psi$ and $\beta^\phi$.
★ Stationarity ($\partial_t\gamma_{ij} = 0, \partial_tK_{ij} = 0$) yields 2 independent equations that fix $A$ and $B$. 
“Traditional” Black-Hole Data

Conformal TT with conformal flatness and maximal slicing

\[ \tilde{\gamma}_{ij} = f_{ij} \text{ (flat)} \]
\[ \tilde{M}^{ij} = 0 \]
\[ K = 0 \]

\[ \Rightarrow \begin{cases} \tilde{\Delta}_L X^i = 0 \Rightarrow \\
\tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \end{cases} \]

Bowen-York solution \([14, 45]\)
Analytic solutions for \(\tilde{A}_{ij}\)
“Traditional” Black-Hole Data

Conformal TT with conformal flatness and maximal slicing

\[
\begin{align*}
\tilde{\gamma}_{ij} &= f_{ij} \text{ (flat)} \\
\tilde{M}^{ij} &= 0 \\
K &= 0
\end{align*}
\]

\[\Rightarrow \left\{ \tilde{\Delta}_L X^i = 0 \Rightarrow \tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \right\}
\]

Bowen-York solution [14, 45]

Analytic solutions for \( \tilde{A}_{ij} \)

Three general solution schemes

Conformal Imaging-[52, 14, 21, 26]

Puncture Method-[47, 17, 15]

Apparent Horizon BC-[62]

Inversion symmetry inner-BC

No inner-BC: singular behavior factored out

Apparent horizon inner-BC
“Traditional” Black-Hole Data

Conformal TT with conformal flatness and maximal slicing

\[ \tilde{\gamma}_{ij} = f_{ij} \text{ (flat)} \]
\[ \tilde{M}^{ij} = 0 \]
\[ K = 0 \]
\[ \tilde{\Delta}_L X^i = 0 \]

\[ \tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}^{ij} \tilde{A}^{ij} = 0 \]

Three general solution schemes

Conformal Imaging-[52, 14, 21, 26]

Puncture Method-[47, 17, 15]

Apparent Horizon BC-[62]

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices of \( \tilde{\gamma}_{ij} = f_{ij} \) and Bowen-York \( \tilde{A}^{ij} \).

– Greg Cook – (WFU Physics)
Black-Hole Data

- The “traditional” black hole initial data approach was motivated by computational convenience, not by any strong physical arguments. Research focused on:
  - methods for solving the Hamiltonian constraint for one or two holes. \([77, 20, 21, 26, 1, 42, 57, 33]\)
  - understanding the physical content (and limitations) of initial data containing one or two holes.\([77, 30, 21, 25]\)
  - finding solutions that represented two black holes in nearly circular orbits.\([22, 2, 57]\)
Black-Hole Data

• The “traditional” black hole initial data approach was motivated by computational convenience, not by any strong physical arguments. Research focused on:
  – methods for solving the Hamiltonian constraint for one or two holes. \([77, 20, 21, 26, 1, 42, 57, 33]\)
  – understanding the physical content (and limitations) of initial data containing one or two holes.\([77, 30, 21, 25]\)
  – finding solutions that represented two black holes in nearly circular orbits.\([22, 2, 57]\)

• Early departures from the “traditional approach” gave up the assumption of conformal flatness so that initial data containing strong waves could be constructed. However the initial data was assumed to be time-symmetric \((K_{ij} = 0)\) or \(\tilde{A}_{ij}\) was found analytically, so only the Hamiltonian constraint had to be solved. \([5, 16]\)
Black-Hole Data

• The “traditional” black hole initial data approach was motivated by *computational convenience*, not by any strong physical arguments. Research focused on:
  – methods for solving the Hamiltonian constraint for one or two holes. [77, 20, 21, 26, 1, 42, 57, 33]
  – understanding the physical content (and limitations) of initial data containing one or two holes.[77, 30, 21, 25]
  – finding solutions that represented two black holes in nearly circular orbits.[22, 2, 57]

• Early departures from the “traditional approach” gave up the assumption of conformal flatness so that initial data containing strong waves could be constructed. However the initial data was assumed to be time-symmetric \((K_{ij} = 0)\) or \(\tilde{A}_{ij}\) was found analytically, so only the Hamiltonian constraint had to be solved. [5, 16]

• Recent work has generalized many of these assumption and requires solving the full set of 4 (or more) coupled, non-linear elliptic PDEs. [51, 6, 50, 38, 41, 56]
Neutron-Star Data[59]

- Single, stationary neutron stars have been studied extensively. The matter is assumed to be in hydrostatic equilibrium. Research has focused on:
  - various numerical methods.[13, 19, 18, 43, 27, 29, 60, 12, 37, 8, 10]
  - rigid and differential rotation.[69, 44, 27]
  - studies with various equations of state.[34, 28]
Single, stationary neutron stars have been studied extensively. The matter is assumed to be in hydrostatic equilibrium. Research has focused on:

- various numerical methods.[13, 19, 18, 43, 27, 29, 60, 12, 37, 8, 10]
- rigid and differential rotation.[69, 44, 27]
- studies with various equations of state.[34, 28]

Binary systems have also been studied. These are usually based on the assumption that the binary is nearly in equilibrium (quasi-equilibrium). Research has focused on:

- various numerical methods.[70, 71, 9, 3, 39, 40]
- corotating (tidally locked) binaries.[9, 3, 4, 48, 65]
- irrotational binaries.[61, 58, 36, 11, 49, 63, 64]
Current Issues

- Current initial data schemes do not represent, with sufficient detail, the physical configurations we are most interested in — astrophysical compact binary inspiral.
Current Issues

• Current initial data schemes do not represent, with sufficient detail, the physical configurations we are most interested in — astrophysical compact binary inspiral.
  – The conformal-flatness assumption does not represent rapidly spinning objects well. It also contributes to unphysical gravity waves in orbiting binaries.\cite{35, 57}
  – The Bowen-York analytic extrinsic curvature does not model the physics of binary inspiral well.\cite{41, 56, 31}
Current Issues

- Current initial data schemes do not represent, with sufficient detail, the physical configurations we are most interested in — astrophysical compact binary inspiral.
  - The conformal-flatness assumption does not represent rapidly spinning objects well. It also contributes to unphysical gravity waves in orbiting binaries.[35, 57]
  - The Bowen-York analytic extrinsic curvature does not model the physics of binary inspiral well.[41, 56, 31]

☆ Boosted Kerr — Use a superposition of analytic black hole metrics and extrinsic curvatures as trial data and solve the constraints to correct them.[51, 50]
Current Issues

• Current initial data schemes do not represent, with sufficient detail, the physical configurations we are most interested in — astrophysical compact binary inspiral.
  – The conformal-flatness assumption does not represent rapidly spinning objects well. It also contributes to unphysical gravity waves in orbiting binaries.[35, 57]
  – The Bowen-York analytic extrinsic curvature does not model the physics of binary inspiral well.[41, 56, 31]

★ Boosted Kerr — Use a superposition of analytic black hole metrics and extrinsic curvatures as trial data and solve the constraints to correct them.[51, 50]
  – This approach lacks well motivated boundary conditions and yields results that are only marginally better than “Bowen-York” methods at representing astrophysical binary inspiral.[56]
Current Issues

- Current initial data schemes do not represent, with sufficient detail, the physical configurations we are most interested in — astrophysical compact binary inspiral.
  - The conformal-flatness assumption does not represent rapidly spinning objects well. It also contributes to unphysical gravity waves in orbiting binaries.[35, 57]
  - The Bowen-York analytic extrinsic curvature does not model the physics of binary inspiral well.[41, 56, 31]

☆ Boosted Kerr — Use a superposition of analytic black hole metrics and extrinsic curvatures as trial data and solve the constraints to correct them.[51, 50]
  - This approach lacks well motivated boundary conditions and yields results that are only marginally better than “Bowen-York” methods at representing astrophysical binary inspiral.[56]

☆ Forced-equilibrium — Enforce a “helical Killing-vector” yielding data with equal amounts of ingoing and outgoing radiation — then subtract away the unphysical ingoing radiation. \((\partial_t \gamma_{ij} = \partial_t K_{ij} = 0)\)[7, 32, 68, 67, 66]
  - Questions about boundary conditions, etc. . .
Quasi-equilibrium — For sufficiently separated binaries, the timescale for orbital decay is much larger than the orbital period. Assume only $\partial_t \tilde{\gamma}_{ij} = \partial_t \text{Tr} K = 0$. 
Current Issues

- Quasi-equilibrium — For sufficiently separated binaries, the timescale for orbital decay is much larger than the orbital period. Assume only $\partial_t \tilde{\gamma}_{ij} = \partial_t \text{Tr} K = 0$.

  - For black holes, use topology and quasi-equilibrium to fix boundary conditions on the constrained data.\cite{38, 41} Leads to overdetermined boundary conditions — only satisfied in true equilibrium.
Current Issues

- Quasi-equilibrium — For sufficiently separated binaries, the timescale for orbital decay is much larger than the orbital period. Assume only $\partial_t \tilde{\gamma}_{ij} = \partial_t \text{Tr} K = 0$.

  - For black holes, use topology and quasi-equilibrium to fix boundary conditions on the constrained data.\cite{38, 41} 
    Leads to overdetermined boundary conditions — only satisfied in true equilibrium.

  - For black holes, use the principle of quasi-equilibrium to derive boundary conditions for the constrained data.\cite{24} 
    Leads to complicated, elliptic boundary conditions. May not be well posed.
Current Issues

★ Quasi-equilibrium — For sufficiently separated binaries, the timescale for orbital decay is much larger than the orbital period. Assume only \( \partial_t \tilde{\gamma}_{ij} = \partial_t \text{Tr} K = 0 \).

- For black holes, use topology and quasi-equilibrium to fix boundary conditions on the constrained data.\[38, 41\] *Leads to overdetermined boundary conditions — only satisfied in true equilibrium.*

- For black holes, use the principle of *quasi-equilibrium* to derive boundary conditions for the constrained data.\[24\] *Leads to complicated, elliptic boundary conditions. May not be well posed.*

- How do we choose \( \tilde{\gamma}_{ij} \)?\[65, 56\]
  ∗ Can we use post-Newtonian metrics to improve the choice of \( \tilde{\gamma}_{ij} \)?
  ∗ Can we use a perturbative evolution to iteratively improve the choice of \( \tilde{\gamma}_{ij} \)?
Quasi-Equilibrium with Black Hole Excision

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \]
\[ K_{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \]
\[ \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{L} \beta)^{ij} \]
\[ \partial_t \tilde{\gamma}_{ij} = 0 \]

\[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \]
\[ \tilde{\Delta} \tilde{L} \beta^i - (\tilde{L} \beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K \]
\[ \tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[ \frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} \tilde{A}_{ij} \tilde{A}^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \]
\[ \partial_t K = 0 \]

\[ \tilde{s}^k \tilde{\nabla}_k \ln \psi \mid_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \mid_S \quad \theta = 0 \]
\[ \beta^i \mid_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i \mid_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i \mid_S + \Omega \tilde{h}^i_j \left( \frac{\partial}{\partial \phi} \right)^j \mid_S & \text{irrotation} \end{cases} \]
\[ J \tilde{s}^i \tilde{\nabla}_i \alpha \mid_S = -\psi^2 (J^2 - JK + \tilde{D}) \alpha \mid_S \quad \mathcal{L}_\zeta \theta = 0 \]
\[ \mathcal{L}_\zeta \dot{\theta} = 0 \]

The only remaining freedom in the system is the choice of the orbital angular velocity \( \Omega \), the initial spatial gauge and the initial dynamical ("wave") content found in \( \tilde{\gamma}_{ij} \), and the initial temporal gauge in \( K \).
References


