Black-Hole Binaries in Quasi-Equilibrium

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Abstract

A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then a complete set of boundary conditions for all initial data variables can be developed. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data.
Motivation

- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, . . .
- Available computed waveforms should increase chance of detecting collision events.
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- General Relativity doesn’t permit true equilibrium for astrophysical binary systems.
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★ Quasi-equilibrium gives us a physical condition to guide us in fixing boundary conditions and data that is not otherwise constrained.
The 3 + 1 Decomposition

Lapse : $\alpha$  
Spatial metric : $\gamma_{ij}$

Shift vector : $\beta^i$  
Extrinsic Curvature : $K_{ij}$

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

$$K_{\mu\nu} = -\frac{1}{2} \gamma^\alpha_{\mu} \gamma^\beta_{\nu} \mathcal{L}_n g_{\alpha\beta}$$
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Constraint equations

$$\bar{R} + K^2 - K_{ij} K^{ij} = 16\pi \rho$$

$$\bar{\nabla}_j (K^{ij} - \gamma^{ij} K) = 8\pi j^i$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[ \bar{R}_{ij} - 2 K_{i\ell} K^{\ell}_{j} + K K_{ij} ight.$$

$$+ \left. 8\pi S_{ij} + 4\pi \gamma_{ij} (S - \rho) \right]$$

$$+ \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell} \bar{\nabla}_j \beta^\ell + K_{j\ell} \bar{\nabla}_i \beta^\ell$$

$S_{\mu\nu} \equiv \gamma^\alpha_{\mu} \gamma^\beta_{\nu} T_{\alpha\beta}$

$j_\mu \equiv -\gamma^\nu_{\mu} n^\alpha T_{\nu\alpha}$

$\rho \equiv n^\mu n^\nu T_{\mu\nu}$

$T_{\mu\nu} = S_{\mu\nu} + 2 n_{(\mu} j_{\nu)} + n_\mu n_\nu \rho$
Degrees of Freedom

**Kinematical variables**
- Lapse $\alpha$ : 1 degree of freedom
- Shift $\beta^i$ : 3 degrees of freedom

**Initial-data variables**
- Metric $\gamma_{ij}$ : 6 degrees of freedom
- Extrinsic curvature $K_{ij}$ : 6 degrees of freedom
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**Decomposition of initial-data variables**[5] (Conformal TT decomp)

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \begin{cases} 
\psi : & 1 \text{ constrained DOF} \\
\tilde{\gamma}_{ij} : & 3 \text{ spatial gauge DOF} \\
\end{cases} \begin{cases}
\tilde{\gamma}_{ij} : & 2 \text{ dynamical DOF}
\end{cases}
\]
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\]

\[
K^{ij} = \psi^{-10} \left[ (\tilde{L}X)^{ij} + \tilde{Q}^{ij} \right] + \frac{1}{3} \gamma^{ij} K
\]

\[
\begin{align*}
\tilde{Q}^{ij} & : \begin{cases} 
\tilde{\nabla}_j \tilde{Q}^{ij} = \tilde{Q}^i_i = 0 \\
2 \text{ dynamical DOF}
\end{cases} \\
(\tilde{L}X)^{ij} & \equiv \tilde{\nabla}^i X^j + \tilde{\nabla}^j X^i - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\nabla}_k X^k \\
X^i & : \begin{cases} 
(\tilde{L}X)^{ij} \equiv \tilde{\nabla}^i X^j + \tilde{\nabla}^j X^i - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\nabla}_k X^k \\
3 \text{ constrained DOF}
\end{cases} \\
K & : 1 \text{ temporal gauge DOF}
\end{align*}
\]
Specifying Initial Data

Freely specified degrees of freedom

\[ \tilde{\gamma}_{ij} \leftarrow \begin{cases} 
(2) \text{ initial dynamical ("wave") content} \\
(3) \text{ initial spatial gauge choices} 
\end{cases} \]

\[ \tilde{Q}^{ij} \leftarrow (2) \text{ initial dynamical ("wave") content} \]

\[ K \leftarrow (1) \text{ initial temporal gauge choice} \]

Freely specified dynamical gauge freedom

\[ \beta^i \leftarrow \text{ how spatial coordinates evolve} \]

\[ \alpha \leftarrow \text{ how time coordinate evolves} \]
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Constrained degrees of freedom

\[ \psi \leftarrow \begin{cases} \text{(1) Hamiltonian constraint} \\ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2 \pi \psi^5 \rho \end{cases} \]

\[ X^i \leftarrow \begin{cases} \text{(3) Momentum constraint} \\ \tilde{\Delta}_{\perp} X^i = \frac{2}{3} \psi^6 \tilde{\nabla}^i K + 8 \pi \psi^{10} j^i \end{cases} \]

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Boundary conditions

The constraints form a set of 4 coupled nonlinear PDEs for (\psi, X^i) that require the specification of boundary conditions at spatial infinity and any interior boundaries.

Freely specified dynamical gauge freedom

\[ \beta^i \leftarrow \text{how spatial coordinates evolve} \]

\[ \alpha \leftarrow \text{how time coordinate evolves} \]
“Traditional” Black-Hole Data

Conformal flatness and maximal slicing

\[
\begin{align*}
\tilde{\gamma}_{ij} &= f_{ij} \text{ (flat)} \\
\tilde{Q}^{ij} &= 0 \\
K &= 0
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
\tilde{\Delta}_L X^i = 0 & \Rightarrow \text{Bowen-York solution} [3] \\
\tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 & \text{Analytic solutions for } \tilde{A}^{ij}
\end{cases}
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Bowen-York solution\textsuperscript{[3]}

Analytic solutions for \(\tilde{A}^{ij}\)

Three general solution schemes

Conformal Imaging-\textsuperscript{[6]}

Inversion symmetry inner-BC

Apparent Horizon BC-\textsuperscript{[11]}

Apparent horizon inner-BC

Puncture Method-\textsuperscript{[4]}

No inner-BC: singular behavior factored out

All methods can produce very general configurations of multiple black holes,
“Traditional” Black-Hole Data

Conformal flatness and maximal slicing

\[ \tilde{\gamma}_{ij} = f_{ij} \text{ (flat)} \]
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\( \Rightarrow \)

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Bowen-York solution \[ ^{[3]} \]
Analytic solutions for \( \tilde{A}_{ij} \)

Three general solution schemes

Conformal Imaging-\[ ^{[6]} \]
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No inner-BC: singular behavior factored out

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for \( \tilde{\gamma}_{ij} \) and Bowen-York \( \tilde{A}^{ij} \).
“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.
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How do we construct improved BH initial data?

We must carefully choose the

- initial dynamical degrees of freedom \([\tilde{\gamma}_{ij} \text{ and } \tilde{Q}^{ij}]\)
- initial temporal and spatial gauge degrees of freedom \([\tilde{\gamma}_{ij} \text{ and } K]\)
- boundary conditions on the constrained degrees of freedom \([\psi \text{ and } X^i]\)

so as to conform to the desired physical content of the initial data.
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How do we construct improved BH initial data?

We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and $\tilde{Q}^{ij}$]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and $K$]
- boundary conditions on the constrained degrees of freedom [in $\psi$ and $X^{i}$]

so as to conform to the desired physical content of the initial data.

- For black holes in quasi-circular orbits, we can use the principle of quasi-equilibrium to guide our choices.
- Quasi-equilibrium is a dynamical concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.
Conformal Thin-Sandwich Decomposition[13]

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \]

\[ K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[ (\tilde{L}/\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3} \gamma^{ij} K \begin{cases} 
\tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij} \quad (\tilde{u}_i^i = 0) \\
\tilde{\alpha} \equiv \psi^{-6} \alpha 
\end{cases} \]

Hamiltonian Const. \[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \]

Momentum Const. \[ \tilde{\Delta}_{\tilde{L}} \beta^i - (\tilde{L}/\beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} = \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K + \tilde{\alpha} \tilde{\nabla}_j \left( \frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16\pi \tilde{\alpha} \psi^{10} j^i \]

\[ \tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{L}/\beta)^{ij} - \tilde{u}^{ij} \right] \]

Constrained vars: \( \psi \) and \( \beta^i \)

Freely specified: \( \tilde{\gamma}_{ij}, \tilde{u}^{ij}, K, \) and \( \tilde{\alpha} \)

\( \tilde{u}^{ij} \) and \( \beta^i \) have a simple physical interpretation, unlike \( \tilde{Q}^{ij} \) and \( X^i \).
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Freely specified : \( \tilde{\gamma}_{ij}, \tilde{u}^{ij}, K, \) and \( \tilde{\alpha} \)

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Quasi-equilibrium \( \Rightarrow \)
\[ \begin{align*}
\tilde{u}^{ij} &= 0 \\
\partial_t K &= 0 \quad (\text{Const. on } \alpha)
\end{align*} \]

Const. Tr(\( K \)) eqn.
\[ \tilde{\nabla}^2 (\alpha \psi) - \alpha \left[ \frac{1}{8} \psi \tilde{R} + \frac{5}{12} \psi^5 K^2 + \frac{7}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} \right. \]
\[ \left. + 2\pi \psi^5 K (\rho + 2S) \right] = \psi^5 \beta^i \tilde{\nabla}_i K \]
Equations of Quasi-Equilibrium

Ham. & Mom. const. eqns. from Conf. TS + Const. Tr($K$) eqn. \[ \Rightarrow \] Eqns. of Quasi-Equilibrium

With $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, and $K = 0$, these equations have been widely used to construct binary neutron star initial data[1, 10, 2, 12].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation. 
  \[ \psi|_{r \to \infty} = 1 \quad \beta^i|_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \to \infty} = 1 \]

- compatible solution of the equations of hydrostatic equilibrium. (\( \Rightarrow \Omega \))
Equations of Quasi-Equilibrium

\[ \text{Ham. \& Mom. const. eqns. from Conf. TS} \]
\[ + \text{Const. } \text{Tr}(K) \text{ eqn.} \implies \text{Eqns. of Quasi-Equilibrium} \]

Binary neutron star initial data require:
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- compatible solution of the equations of hydrostatic equilibrium. \((\implies \Omega)\)

Binary black hole initial data require:
- a means for choosing the angular velocity of the orbit \(\Omega\).
  \[
  \star \text{ with excision, inner boundary conditions are needed for } \psi, \beta^i, \text{ and } \tilde{\alpha}.
  \]

Gourgoulhon, Grandclément, & Bonazzola\[8, 9\]: Black-hole binaries with \(\tilde{\gamma}_{ij} = f_{ij}, \tilde{u}^{ij} = 0, K = 0\), “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require constraint violating regularity condition imposed on inner boundaries!
Why does the GGB approach fail?

• Inversion-symmetry demands $\tilde{\alpha} = 0$ & $\tilde{K} = 0$ on the inner boundary.

• It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{L}/\beta)^{ij} - \tilde{u}^{ij} \right]$$
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How do we proceed?

- Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & $K$.

- Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.
Constructing Regular Binary Black Hole QE ID

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How do we proceed?

• Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & $K$.

★ Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.

Approach

• Demand that the excision (inner) boundary be an apparent horizon.

• Demand that the apparent horizon be in quasi-equilibrium.
The Inner Boundary

\[ s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|} \]
\[ h_{ij} \equiv \gamma_{ij} - s_i s_j \]
\[ k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu) \]
\[ \dot{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu) \]

Extrinsic curvature of \( S \) embedded in spacetime

\[ \Sigma_{\mu\nu} \equiv -\frac{1}{2} h^\alpha_{\mu} h^\beta_{\nu} \mathcal{L}_{\dot{k}} g_{\alpha\beta} \]
\[ \dot{\Sigma}_{\mu\nu} \equiv -\frac{1}{2} h^\alpha_{\mu} h^\beta_{\nu} \mathcal{L}_{\dot{k}} g_{\alpha\beta} \]
Extrinsic curvature of \( S \) embedded in \( \Sigma \)

\[
H_{ij} \equiv -\frac{1}{2} h^k_i h^\ell_j \mathcal{L}_s \gamma_{k\ell}
\]

Projections of \( K_{ij} \) onto \( S \)

\[
J_{ij} \equiv h^k_i h^\ell_j K_{k\ell}
\]
\[
J_i \equiv h^k_i s^\ell K_{k\ell}
\]
\[
J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}
\]

Extrinsic curvature of \( S \) embedded in \( \Sigma \)

\[
\sum_{\mu\nu} \equiv -\frac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_k g_{\alpha\beta}
\]

\[
\dot{\sum}_{\mu\nu} \equiv -\frac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_k g_{\alpha\beta}
\]
**The Inner Boundary**

Extrinsic curvature of $S$ embedded in $\Sigma$

\[ H_{ij} \equiv -\frac{1}{2} h_i^k h_j^\ell L_s \gamma_{k\ell} \]

Projections of $K_{ij}$ onto $S$

\[ J_{ij} \equiv h_i^k h_j^\ell K_{k\ell} \]
\[ J_i \equiv h_i^k s^\ell K_{k\ell} \]
\[ J \equiv h^{ij} J_{ij} = h^{ij} K_{ij} \]

Expansion of null rays

\[ \sigma \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (J + H) \]

Shear of null rays

\[ \sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \sigma \]
\[ \dot{\sigma} \equiv h^{ij} \dot{\Sigma}_{ij} = \frac{1}{\sqrt{2}} (J - H) \]

\[ \dot{\sigma}_{ij} \equiv \dot{\Sigma}_{ij} - \frac{1}{2} h_{ij} \dot{\sigma} \]
AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $S$ is a (MOTS):
   \[ \sigma = 0 \]

2. The inner boundary $S$ doesn’t move:
   \[ \mathcal{L}_{\zeta} \tau = 0 \text{ and } \hat{\nabla}_i \mathcal{L}_{\zeta} \tau \equiv h_i^j \hat{\nabla}_j \mathcal{L}_{\zeta} \tau = 0 \]
   \[ t^\mu = \alpha n^\mu + \beta^\mu \quad \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu \]
   \[ \beta_\perp \equiv \beta^i s_i \]

3. The inner boundary $S$ remains a MOTS[7]:
   \[ \mathcal{L}_{\zeta} \sigma = 0 \text{ and } \mathcal{L}_{\zeta} \dot{\sigma} = 0 \]

4. The horizons are in quasi-equilibrium:
   \[ \sigma_{ij} = 0 \text{ and no matter is on } S \]
Evolution of the Expansions

\[ \mathcal{L}_{\zeta \sigma} = \frac{1}{\sqrt{2}} \left[ \sigma \left( \sigma + \frac{1}{2} \dot{\sigma} - \frac{1}{\sqrt{2}} K \right) + \mathcal{E} \right] (\beta_\perp + \alpha) \]

\[ + \frac{1}{\sqrt{2}} \left[ \sigma \left( \frac{1}{2} \sigma - \frac{1}{2} \dot{\sigma} - \frac{1}{\sqrt{2}} K \right) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu \dot{k}^\nu \right] (\beta_\perp - \alpha) \]

\[ + \sigma s^i \tilde{\nabla}_i \alpha, \]

\[ \mathcal{L}_{\zeta \dot{\sigma}} = -\frac{1}{\sqrt{2}} \left[ \dot{\sigma} \left( \sigma + \frac{1}{2} \dot{\sigma} - \frac{1}{\sqrt{2}} K \right) + \dot{\mathcal{E}} \right] (\beta_\perp - \alpha) \]

\[ - \frac{1}{\sqrt{2}} \left[ \dot{\sigma} \left( \frac{1}{2} \dot{\sigma} - \frac{1}{2} \sigma - \frac{1}{\sqrt{2}} K \right) + \dot{\mathcal{D}} + 8\pi T_{\mu\nu} k^\mu \dot{k}^\nu \right] (\beta_\perp + \alpha) \]

\[ - \dot{\sigma} s^i \tilde{\nabla}_i \alpha, \]

\[ \mathcal{D} \equiv h^{ij} (\hat{\nabla}_i + J_i) (\hat{\nabla}_j + J_j) - \frac{1}{2} \hat{R} \]

\[ \dot{\mathcal{D}} \equiv h^{ij} (\hat{\nabla}_i - J_i) (\hat{\nabla}_j - J_j) - \frac{1}{2} \hat{R} \]

\[ \mathcal{E} \equiv \sigma_{ij} \dot{\sigma}^{ij} + 8\pi T_{\mu\nu} k^\mu \dot{k}^\nu \]

\[ \dot{\mathcal{E}} \equiv \dot{\sigma}_{ij} \dot{\sigma}^{ij} + 8\pi T_{\mu\nu} \dot{k}^\mu \dot{k}^\nu \]

Incorporates the constraint and evolution equations of GR, the Gauss–Codazzi–Ricci equations governing the embedding of \( S \) in the spatial hypersurface, and the demand that \( S \) remain at a constant coordinate location. \textit{These equations incorporate no assumption of quasi-equilibrium.}

Terms that vanish because we demand \( S \) be a MOTS, remain a MOTS, or because we demand the horizon to be in equilibrium.

– Greg Cook – (WFU Physics)
\[ \sigma = 0 \]

\[ 0 = D(\beta_{\perp} - \alpha), \]

\[ \dot{s}^i \tilde{\nabla}_i \alpha = -\frac{1}{\sqrt{2}} \left[ \dot{\sigma}(\dot{\sigma} - \frac{1}{\sqrt{2}} K) + \dot{\sigma}_{ij} \dot{\sigma}^{ij} \right] (\beta_{\perp} - \alpha) \]

\[ -\frac{1}{\sqrt{2}} \left[ \dot{\sigma}(\frac{1}{2} \dot{\sigma} - \frac{1}{\sqrt{2}} K) + \dot{D} \right] (\beta_{\perp} + \alpha). \]

\[ \tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \]

\[ \beta^i = \alpha \psi^{-2} \tilde{s}^i + B^i_{\parallel} \]

\[ J \tilde{s}^i \tilde{\nabla}_i \alpha = -\psi^2 (J^2 - JK + \tilde{D}) \alpha \]

\[ h_{ij} \equiv \psi^4 \tilde{h}_{ij} \]

\[ s^i \equiv \psi^{-2} \tilde{s}^i \]

\[ B^i_{\parallel} s_i = 0 \]

\[ \tilde{D} \equiv \psi^{-4} [\tilde{h}^{ij} (\tilde{\nabla}_i - J_i)(\tilde{\nabla}_j - J_j) - \frac{1}{2} \tilde{R} + 2 \tilde{\nabla}^2 \ln \psi] \]

\[ [\tilde{\nabla} & \tilde{R} are compatible with \tilde{h}_{ij}] \]

The conditions of quasi-equilibrium yield boundary conditions for 3 of the 5 constrained variables (\( \psi, \alpha, \beta_{\perp} \)). The remaining two conditions are contained in the definition of \( \beta^i_{\parallel} \). This freedom is necessary to prescribe the spin of the black hole.
Defining the Spin of the Black Hole

The spin parameters $\beta_i^i$ can be defined by demanding that the MOTS be a Killing horizon. The time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^\mu \propto (n^\mu + s^\mu) \quad \implies \quad k^\mu = \left[ 1, \alpha s^i - \beta^i \right]$$

This vector $k^\mu$ is null for any choice of $\beta^i$.

In the frame where a black hole is not spinning, the null time vector has components $t^\mu = [1, \vec{0}]$.

### Corotating Holes
Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^\mu = \left[ 1, \alpha s^i - \beta^i \right] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \quad \Rightarrow \quad \beta^i_\parallel = 0$$

### Irrotational Holes
Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^\mu = \left[ 1, \alpha s^i - \beta^i \right] = [1, -\Omega(\partial/\partial \phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left( \frac{\partial}{\partial \phi} \right)^i \quad \Rightarrow \quad \beta^i_\parallel = \Omega \left( \frac{\partial}{\partial \phi} \right)^i$$
Summary of QE Formalism

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{L} \beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0 \]

\[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \]
\[ \tilde{\Delta}_{\tilde{L} \beta}^i - (\tilde{\nabla}_j \ln \alpha \psi)^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K \]
\[ \tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[ \frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} \tilde{A}_{ij} \tilde{A}^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0 \]

\[ s^k \tilde{\nabla}_k \ln \psi \bigg|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \bigg|_S \quad \sigma = 0 \]
\[ \beta^i \bigg|_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i \bigg|_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i \bigg|_S + \Omega \tilde{h}^i_j \left( \frac{\partial}{\partial \phi} \right)^j \bigg|_S & \text{irrotation} \end{cases} \]
\[ J \tilde{s}^i \tilde{\nabla}_i \alpha \bigg|_S = -\psi^2 (J^2 - JK + \tilde{D}) \alpha \bigg|_S \quad \mathcal{L}_\zeta \sigma = 0 \quad \sigma_{ij} = 0 \]

\[ \psi \bigg|_{r \to \infty} = 1 \quad \beta^i \bigg|_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^i \quad \alpha \bigg|_{r \to \infty} = 1 \]

The only remaining freedom in the system is the choice of the orbital angular velocity, the initial spatial and temporal gauge, and the initial dynamical ("wave") content found in \( \Omega, \tilde{\gamma}_{ij} \) and \( K \).
The Orbital Angular Velocity

• For a given choice of $\tilde{\gamma}_{ij}$ and $K$, we are still left with a family of solutions parameterized by the orbital angular velocity $\Omega$.

• Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of $\Omega$ to correspond to a system in quasi-equilibrium.

GGB[8, 9] have suggested a way to pick the quasi-equilibrium value of $\Omega$: 
The Orbital Angular Velocity

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GGB[8, 9] have suggested a way to pick the quasi-equilibrium value of $\Omega$:

$\Omega$ is chosen as the value for which the ADM energy $E_{ADM}$ equals the Komar mass $M_K$.

Komar mass
$$M_K = \frac{1}{4\pi} \int_\infty \gamma^{ij} \left( \nabla_i \alpha - \beta^k K_{ik} \right) d^2 S_j$$
Acceptable definition of the mass only for stationary spacetimes.

ADM energy
$$E_{ADM} = \frac{1}{16\pi} \int_\infty \gamma^{ij} \nabla_k (G^k_i - \delta^k_i G) d^2 S_j$$
Acceptable definition of the mass for arbitrary spacetimes.
$$G_{ij} \equiv \gamma_{ij} - f_{ij}$$
Remaining Issues

- Are there solutions satisfying the AH/QE boundary conditions?
  - Stationary BH solutions.
  - Time-symmetric & inversion-symmetric BH solutions.
  - Others?
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  - The problem is with inversion symmetry, not conformal flatness. Only time-symmetric or stationary solutions of the QE equations can be inversion-symmetric.
  - Known stationary solutions on maximal slices are inversion-symmetric, so maximal slicing may be a problem.
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• How do we choose $\tilde{\gamma}_{ij}$ and $K$?
  - Conformal flatness & $K \sim 1/r^2$?
  - Use “superposition” of Kerr?
  - Use post-Newtonian metrics (no radiation reaction)?
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- Do numerical solutions exist?
  - H. Pfeiffer and I are working on it.
  - The lapse boundary condition is difficult.
  - We are currently seeing some “strange” behavior which is either a bug or something very interesting.
References


