Higher Dimensional Black Holes

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Kaluza-Klein, String Theory and other theories suggest that Space-time may have more than 4 dimensions. The simplest example is:

\[ R^{3,1} \times S^1 \] - one additional dimension
d – the number of dimensions is the only free parameter in classical GR. It is interesting, therefore, from a pure theoretical point of view to explore the behavior of the theory when we vary this parameter (Kol, 04).

<table>
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<th>2d</th>
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<th>4d</th>
<th>5d</th>
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<td>trivial</td>
<td>almost trivial</td>
<td>difficult</td>
<td>you ain’t seen anything yet</td>
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What is the structure of Higher Dimensional Black Holes?

**Black String**

$\mathbb{R}^4 \times S^1$

Horizon topology $S^2 \times S^1$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d^2\Omega + dz^2$$

Asymptotically flat 4d space-time $\times S^1$
Or a black hole?

Locally:

\[ ds^2 = -(1 - \frac{(8/3\pi)M}{R^2})dt^2 + (1 - \frac{(8/3\pi)M}{R^2})^{-1}dR^2 + R^2 d^2\Omega^{(3)} \]

R is the 5D distance from the origin

r is the 4D distance from the origin

Horizon topology
\( S^3 \)

Asymptotically flat 4d space-time x \( S^1 \)

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d^2\Omega + dz^2 \]
Black String

Black Hole
A single dimensionless parameter

\[
\mu \equiv \frac{G_N m}{L^{d-3}}
\]

Black hole
A single dimensionless parameter

\[ \mu \equiv \frac{G_N m}{L^{d-3}} \]
What Happens in the inverse direction - a shrinking String?

μ = \frac{G_N m}{L^{d-3}}

Gregory & Laflamme (GL): the string is unstable below a certain mass.

Compare the entropies (in 5d):

S_{bh} \sim \mu^{3/2} \text{ vs. } S_{bs} \sim \mu^2

Expect a phase transition
Horowitz-Maeda (‘01): Horizon doesn’t pinch off!

The end-state of the GL instability is a stable non-uniform string.

\[\mu_{\text{non-uniform}} > \mu_{\text{uniform}} \quad S_{\text{non-uniform}} > S_{\text{uniform}}\]

Perturbation analysis around the GL point \([\text{Gubser 5d ‘01}]\) the non-uniform branch emerging from the GL point cannot be the end-state of this instability.

Dynamical: no signs of stabilization (5d) \([\text{CLOPPV ‘03}]\).
This branch non-perturbatively (6d) \([\text{Wiseman ‘02}]\).
Dynamical Instability of a Black String

\[ \lambda = \frac{r_{\text{max}}}{r_{\text{min}}} - 1 \] / 2

Choptuik, Lehner, Olabarrieta, Petryk, Pretorius & Villegas 2003
Non-Uniform Black String Solutions
(Wiseman, 02)
Objectives:

• Explore the structure of higher dimensional spherical black holes.
• Establish that a higher dimensional black hole solution exists in the first place.
• Establish the maximal black hole mass.
• Explore the nature of the transition between the black hole and the black string solutions.
The Static Black hole Solutions

\[ ds^2 = -A^2 dt^2 + e^{2B} (dr^2 + dz^2) + e^{2C} r^2 d\Omega_2^2 , \]

\[ r = \rho \sin \chi, \quad z = \rho \cos \chi, \]

\[ ds^2 = -A^2 dt^2 + e^{2B} (d\rho^2 + \rho^2 d\chi^2) + e^{2C} \rho^2 \sin^2 \chi d\Omega_2^2 , \]
Figure 1: A spacelike slice of the black-hole spacetime. (a) In the \( \{r, z\} \) plane the black hole’s horizon is a curve with a spherical \( S^3 \) topology. (b) There is a conformal freedom to transform the domain to \( \{(r, z) : |z| \leq L, r^2 + z^2 \geq \rho_h^2\} \). By fixing \( \rho_h/L \) the domain is uniquely specified [21].
Equations of Motion

\[
\Delta A + \frac{2 \partial \xi A}{\rho^2} \left( -\xi + (1 - \xi^2) \partial \xi C \right) + 2 \partial \rho A \left( \frac{1}{\rho} + \partial \rho C \right) = 0,
\]

\[
\Delta B + \frac{2 \partial \xi A}{A \rho^2} \left( \xi - (1 - \xi^2) \partial \xi C \right) + \frac{2 \partial \xi C}{A} \left( \xi - \frac{1}{2} (1 - \xi^2) \partial \xi C \right) + \frac{\rho^2}{\rho^2 (1 - \xi^2)} \left( 1 - e^{2(B-C)} \right) = 0,
\]

\[
\Delta C - \frac{\partial \xi A}{A \rho^2} \left( \xi - (1 - \xi^2) \partial \xi C \right) - \frac{4 \partial \xi C}{A} \left( \xi - \frac{1}{2} (1 - \xi^2) \partial \xi C \right) + \frac{\rho^2}{\rho^2 (1 - \xi^2)} \left( 1 - e^{2(B-C)} \right) = 0.
\]

\[
\xi \equiv \cos(\chi) \equiv z / \sqrt{r^2 + z^2}
\]

\[
\Delta \psi = -c \psi
\]
Poor man’s Gravity –  
The Initial Value Problem (Sorkin & TP 03) 
Consider a moment of time symmetry: 
\[ ds^2 = \psi^4 (dr^2 + dz^2 + r^2 \, d\Omega^2) \]
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} + \tilde{\rho} \psi^{-8} = 0 \]

There is a Bh-Bstr transition 

Higher Dim Black holes exist?
Proper distance

Log($\mu - \mu_c$)
A similar behavior is seen in 4D

Apparent horizons: From a simulation of bh merger Seidel & Brügmann
\[ \lambda = \left( \frac{r'_{\text{max}}}{r'_{\text{min}}} - 1 \right) / 2 \]

\[ \lambda' = \left( \frac{r'_{\text{max}}}{r'_{\text{min}}} - 1 \right) \]
The Anticipated phase diagram

We need a “good” order parameter allowing to put all the phases on the same diagram. [Scalar charge]
Asymptotic behavior

At $r \to \infty$:

- The metric becomes $z$-independent as: $[ \sim \exp(-r/L) ]$
- Newtonian limit

\[
\begin{align*}
g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta} & h_{\alpha\beta} < 1 \\
\bar{h}_{\alpha\beta} &= h_{\alpha\beta} - \frac{1}{2} h_{\gamma} \eta_{\alpha\beta} & \partial^{\alpha} \bar{h}_{\alpha\beta} = 0 \\
\Delta \bar{h}_{\alpha\beta} &= -16\pi G_d T_{\alpha\beta}
\end{align*}
\]
Asymptotic Charges:

The mass:
\[ m \equiv \int T_{00} d^{d-1}x \]

The Tension:
\[ \tau \equiv -\int T_{zz} d^{d-1}x/L \]

\[ ds^2 = -e^{2A} dt^2 + e^{2\phi} dz^2 + e^{2C} (dr^2 + r^2 d\Omega_{d-3}^2) \]

Asymptotically:
\[ A \approx \frac{a}{r^{d-4}} + o\left(\frac{1}{r^{d-3}}\right) \]
\[ \phi \approx \frac{b}{r^{d-4}} + o\left(\frac{1}{r^{d-3}}\right) \]

\[ \begin{pmatrix} m \\ \tau \end{pmatrix} = \frac{1}{k_d} \begin{pmatrix} d-3 & -1 \\ 1 & -(d-3) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \]
\[ k_d \equiv \frac{8\pi}{\Omega_{d-3}} \]
The asymptotic coefficient $b$ determines the length along the $z$-direction.

$$b = k_d [(d-3)m - \tau L]$$

The mass opens up the extra dimension, while tension counteracts.

For a uniform $B_{str}$: both effects cancel:

But not for a BH:

Archimedes for BHs
Smarr’s formula (Integrated First Law)

\[ I(\beta, L) = -\beta F = \frac{1}{16\pi G_d} \int dV_d R + \frac{1}{8\pi G_d} \int_{\partial} dV_{d-1} [K - K^0] \]

\[ dI = \frac{\partial I}{\partial \beta} d\beta + \frac{\partial I}{\partial L} dL \]

Together with \( F = m - TS \)

We get \( dm = TdS + \tau dL \)

\[ L \rightarrow (1 + \varepsilon)L \quad [m] = L^{d-3} \quad [Area] = L^{d-2} \]

\[ (d - 3)m = (d - 2)TS + \tau L \]

This formula associates quantities on the horizon (T and S) with asymptotic quantities at infinity (a). It will provide a strong test of the numerics.

\[ TS = \frac{1}{8\pi G_N} \quad AK = \frac{(d - 4)L}{k_d G_N} a \]
Numerical Solution
Sorkin Kol & TP
Numerical Convergence I:
Numerical Convergence II:
Numerical Test I (constraints):
Numerical Test II (The BH Area and Smarr’s formula):
Analytic expressions
Gorbonos & Kol 04

eccentricity

e = -3.3 \times 10^{-5} + 2.89 x^4

distance

Ellipticity

“Archimedes”
A Possible Bh Bstr Transition?

Is there intersection at $x=0.26$ and $\mu_3=0.082$
Anticipated phase diagram [Kol '02]

- Uniform Bstr
- Non-uniform Bstr
- Black hole

Merger point

Kudoh & Wiseman '03
ES,Kol,Piran '03

Gubser 5d ,'01
Wiseman 6d ,'02

d=10 a critical dimension
The phase diagram:

**Gubser**: First order uniform-nonuniform black strings phase transition. Explosions, cosmic censorship?

Universal? **Vary d!** E. Sorkin 2004

**Motivations**: (1) Kol’s critical dimension for the BH-BStr merger (d=10) (2) Problems in numerics above d=10

For $d^{*} > 13$, a sudden change in the order of the phase transition. It becomes smooth.
$$\mu_c \propto \gamma^d$$

$$\left( \mu = \frac{G_N m}{L^{d-3}} \right)$$

with $\gamma = 0.686$

The deviations of the calculated points from the linear fit are less than $2.1\%$
Entropies:

\[ S_{BH}^{(0)} = \frac{1}{4G_d} \Omega_{d-2} \left[ \frac{16\pi m}{(d-2)\Omega_{d-2}} \right]^{\frac{d-2}{d-3}} \]
\[ \mu^{(0)} = \frac{1}{16\pi} \frac{\Omega_{d-3}^{d-3}}{\Omega_{d-2}^{d-4}} \frac{(d-3)^{(d-3)(d-3)}}{(d-3)^{(d-2)(d-4)}} \]

Corrected BH:
Harmark ‘03
Kol&Gorbonos

\[ S_{BH}^{(1)} = S_{BH}^{(0)} \left[ 1 + \frac{\zeta (d-3)16\pi m}{2(d-2)\Omega_{d-2}} + O(\mu^2) \right] \]
\[ S_{BH}^{(1)} (\mu^{(1)}) = S_{Bstr} (\mu^{(1)}) \]

\[ S_{Bstr} = \frac{1}{4G_d} \Omega_{d-3} \left[ \frac{16\pi m}{(d-3)\Omega_{d-3}} \right]^{\frac{d-3}{d-4}} \]

for a given mass the entropy of a caged BH is larger.
The curves intersect at \( d \sim 13 \). This suggests that for \( d > 13 \), a BH is entropically preferable over the string only for \( \mu < \mu_c \). A hint for a “missing link” that interpolates between the phases.
A comparison between a uniform and a non-uniform String: Trends in mass and entropy

\[ \frac{\delta \mu}{\mu} = \eta_1 \lambda^2 + \ldots \quad ; \quad \frac{S_{\text{non-uniform}}}{S_{\text{uniform}}} = 1 + \sigma_2 \lambda^4 \]

For \(d > 13\) the non-uniform string is less massive and has a higher entropy than the uniform one. A smooth decay becomes possible.
Above $d^*=13$ the unstable GL string can decay into a non-uniform state continuously.

- **Uniform Bstr**
- **Perturbative Non-uniform Bstr**
- **BHs**
- **filling the blob**

![Diagram showing transitions between different states](image)

- a smooth merger
- discontinuous
- a new phase, disconnected
Outlook

Continue the non-uniform phase to the non-linear regime.

Check time-evolution: does the evolution stabilize?

Is the smooth p.t. general: more extra dimensions, other topologies. Would the critical dimension become smaller than d=10?
• We have demonstrated the existence of BH solutions.
• Indication for a BH-Bstr transition.
• The global phase diagram depends on the dimensionality of space-time
Open Questions

- Non uniqueness of higher dim black holes
- Topology change at BH-Bstr transition
- Cosmic censorship at BH-Bstr transition
- Thunderbolt (release of L=c^5/G)
- Numerical-Gravitostatics
- Stability of rotating black strings