Abstract

We will take a detailed look at the issues involved in setting the spin of a black hole during the construction of initial data. Theory seems to provide sufficient freedom that any physically allowed spin can be chosen. The more difficult issue is finding a way to choose the specific spin that we want. I will present some new results that explore the spins of individual black holes in both corotating and irrotational binary systems.
Motivation

- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, TAMA, . . .
- Available computed waveforms should increase chance of detecting collision events.

Quasiequilibrium Binary Data

- General Relativity doesn’t permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a corotating reference frame in which the binary appears to be at rest.

★ Quasiequilibrium gives us a physical condition to guide us in fixing boundary conditions and data that is not otherwise constrained.
The 3 + 1 Decomposition

Lapse : $\alpha$  
Spatial metric : $\gamma_{ij}$
Shift vector : $\beta^i$  
Extrinsic Curvature : $K_{ij}$
Time vector : $t^\mu = \alpha \gamma^\mu + \beta^\mu$

ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$
$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$
$K_{\mu\nu} = -\frac{1}{2} \gamma_\mu \gamma_\nu \mathcal{L}_n g_{\alpha\beta}$

Constraint equations

$\bar{R} + K^2 - K_{ij} K^{ij} = 16\pi \rho$
$\nabla_j \left( K^{ij} - \gamma^{ij} K \right) = 8\pi j^i$

Evolution equations

$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$
$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[ \bar{R}_{ij} - 2K_{i\ell} K_{j}^\ell + K K_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij} (S - \rho) \right]$
+ $\beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell} \bar{\nabla}_j \beta^\ell + K_{j\ell} \bar{\nabla}_i \beta^\ell$

$\gamma_{\mu\nu} \equiv \gamma_\mu \gamma_\nu T^{\alpha\beta}$
$j_\mu \equiv -\gamma_\nu n_\mu T_{\nu\alpha}$
$\rho \equiv n_\mu n_\nu T_{\mu\nu}$
$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu} j_{\nu)} + n_\mu n_\nu \rho$
Conformal Thin-Sandwich Decomposition

\[
\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \\
K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[ (\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K
\]

Hamiltonian Const.
\[
\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho
\]

Momentum Const.
\[
\tilde{\nabla}_j (\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} = \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K + \tilde{\alpha} \tilde{\nabla}_j \left( \frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16\pi \tilde{\alpha} \psi^{10} j^i
\]

Const. Tr(\(K\)) eqn.
\[
\tilde{\nabla}^2 (\psi^7 \tilde{\alpha}) - (\psi^7 \tilde{\alpha}) \left[ \frac{1}{8} \psi \tilde{R} + \frac{5}{12} \psi^5 K^2 + \frac{7}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} - \psi^5 \beta^i \tilde{\nabla}_i K \right]
= -2\pi \psi^5 K (\rho + 2S) - \psi^5 \partial_t K
\]

\[
\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]
\]

Constrained vars: \(\psi, \beta^i,\) and \(\tilde{\alpha} \equiv \psi^{-6} \alpha\)

Freely specified:
\[
\tilde{\gamma}_{ij}, \quad \tilde{u}^{ij} \equiv -\partial_t \tilde{\gamma}^{ij}
\]

Quasiequilibrium \(\Rightarrow\)
\[
\begin{cases}
\partial_t \tilde{\gamma}^{ij} = 0 \\
\partial_t K = 0
\end{cases}
\]
Measuring & Setting the Spin

• Spin is only rigorously defined at spatial/null infinity.

• Must use *quasi-local* definition: e.g. Brown & York\cite{2} or Ashtekar & Krishnan\cite{1}

\[
S = -\frac{1}{8\pi} \int_{BH} K_{ij} \xi^i s^j \sqrt{h} d^2x
\]

• Spin is fixed primarily by value of $K_{ij}$: suggests using the shift at the black hole to fix the spin.

• If we assume that the black hole is in approximate equilibrium (*isolated*), then we can connect the shift vector to the null congruence that defines the black hole’s horizon.
The Inner Boundary

Extrinsic curvature of $S$ embedded in spacetime

$$\Sigma_{\mu\nu} \equiv \frac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_k g_{\alpha\beta}$$

$$\dot{\Sigma}_{\mu\nu} \equiv \frac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_k g_{\alpha\beta}$$

Extrinsic curvature of $S$ embedded in $\Sigma$

$$H_{ij} \equiv \frac{1}{2} h^k_i h^\ell_j \mathcal{L}_s \gamma_{k\ell}$$

Projections of $K_{ij}$ onto $S$

$$J_{ij} \equiv h^k_i h^\ell_j K_{k\ell}$$

$$J_i \equiv h^k_i s^\ell K_{k\ell}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (H - J)$$

$$\theta' \equiv h^{ij} \dot{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H + J)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$$

$$\dot{\sigma}_{ij} \equiv \dot{\Sigma}_{ij} - \frac{1}{2} h_{ij} \theta'$$

$$s_i \equiv \frac{\vec{\nabla}_i \tau}{|\vec{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\dot{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

$$\Sigma_{ij} \equiv \frac{1}{\sqrt{2}} (H_{ij} - J_{ij})$$

$$\dot{\Sigma}_{ij} \equiv -\frac{1}{\sqrt{2}} (H_{ij} + J_{ij})$$

Extrinsic curvature of $S$ embedded in $\Sigma$

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Shear of null rays

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AH and QE Conditions on the Inner Boundary

The quasiequilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $S$ is a (MOTS):
   $\theta = 0$

2. The horizons are in quasiequilibrium:
   $\sigma_{ij} = 0$ and no matter is on $S$

   Raychaudhuri’s equation implies that MOTS initially evolves along $k^\mu$.
   $$\mathcal{L}_k \theta = \frac{1}{2} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu = 0$$

3. The time evolution vector is null on $S$:
   $t^\mu k_\mu|_S = 0$
   $$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$
   $$t^\mu = \alpha n^\mu + \beta^\mu$$

   $\alpha|_S = \beta^i s_i|_S \equiv \beta_\perp|_S$
AH/Quasiequilibrium Boundary Conditions

\[ \theta = \frac{\psi^{-2}}{\sqrt{2}} \left[ \tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4 \tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right] \]

\[ \sigma_{ij} = \frac{1}{\sqrt{2}} \left( H_{ij} - \frac{1}{2} h_{ij} H \right) \left( 1 - \frac{\beta_\perp}{\alpha} \right) \]

\[ - \frac{1}{\sqrt{2}} \frac{\psi^4}{\alpha} \left\{ \tilde{D}_{i(\beta_\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_{k(\beta_\parallel l)} - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{j\ell} \tilde{u}^{k\ell} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{k\ell} \tilde{u}^{k\ell}] \right\} \]
AH/Quasiequilibrium Boundary Conditions

$$
\theta = \frac{\psi^{-2}}{\sqrt{2}} \left[ \tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4 \tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right]
$$

$$
\sigma_{ij} = \frac{1}{\sqrt{2}} \frac{1}{(H_{ij} - \frac{1}{2} h_{ij} H)} \left( 1 - \frac{\beta_\perp}{\alpha} \right) 
- \frac{1}{\psi^4} \alpha \left\{ \tilde{D}_{(i\beta_\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{j\ell} \tilde{u}^{k\ell} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{k\ell} \tilde{u}^{k\ell}] \right\}
$$

$$
\tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)
$$

$$
\beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_{\parallel}^i
$$

$$
0 = \tilde{D}_{(i\beta_\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k
$$
AH/Quasiequilibrium Boundary Conditions

\[ \theta = \frac{\psi^{-2}}{\sqrt{2}} \left[ \tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4 \tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right] \]

\[ \sigma_{ij} = \frac{1}{\sqrt{2}} (H_{ij} - \frac{1}{2} h_{ij} H) \left( 1 - \frac{\beta_{\perp}}{\alpha} \right) \]

\[ - \frac{1}{\sqrt{2} \alpha} \left\{ \tilde{D}_{(i \beta || j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\perp}^k - \frac{1}{2} \left[ \tilde{h}_{ik} \tilde{h}_{j\ell} \tilde{u}^{k\ell} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{k\ell} \tilde{u}^{k\ell} \right] \right\} \]

\[ \tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \]

\[ \beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_{||}^i \]

\[ 0 = \tilde{D}_{(i \beta || j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\perp}^k \]

\[ \partial_t \ln \psi = \frac{1}{4} \left[ \tilde{D}_k \beta_{\perp}^k + 4 \beta_{\perp}^k \tilde{D}_k \ln \psi - \frac{1}{2} \tilde{h}_{k\ell} \tilde{u}^{k\ell} + \sqrt{2} \theta - (\alpha - \beta_{\perp}) H \right] \]
Defining the Spin of the Black Hole

The spin parameters $\beta^i_\parallel$ can be defined by demanding that the time vector associated with quasiequilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^\mu \propto (n^\mu + s^\mu) \implies k^\mu = \left[ 1, \alpha s^i - \beta^i \right]$$

This vector $k^\mu$ is null for any choice of $\alpha$ & $\beta^i$.

In the frame where a black hole is not spinning, the null time vector has components $t^\mu = [1, \vec{0}]$.

### Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^\mu = \left[ 1, \alpha s^i - \beta^i \right] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta^i_\parallel = 0$$

### Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \phi}$$

$$k^\mu = \left[ 1, \alpha s^i - \beta^i \right] = [1, -\Omega_0 (\partial / \partial \phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega_0 \left( \frac{\partial}{\partial \phi} \right)^i \implies \beta^i_\parallel = \Omega_0 \xi^i$$

$$\xi^i \approx \left( \frac{\partial}{\partial \phi} \right)^i \quad & \quad \tilde{D}(i \xi_j) - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \xi^k = 0$$
Corotating; Maximal Slice; Comparison; $E_b/\mu$ vs $J/\mu m$

![Graph showing $E_b/\mu$ vs $J/\mu m$ for different models (MS, HKV, EOB - 3PN, EOB - 2PN, EOB - 1PN).](image)
Irrotational; Maximal Slice; Comparison; $E_b/\mu$ vs $J/\mu m$
Maximal Slice; Comparison of ISCO; \( \frac{E_b}{M_{irr}} \) vs \( \Omega M_{irr} \)
Maximal Slice; Comparison of ISCO; $E_b/M_{\text{irr}}$ vs $J/M_{\text{irr}}^2$
Maximal Slice; Comparison of ISCO; $J/M_{irr}^2$ vs $\Omega M_{irr}$

[Diagram showing comparisons of different models (CO: QE, CO: HKV-GGB, CO: PN EOB, CO: PN standard, IR: QE, IR: IVP conf, IR: PN EOB, IR: PN standard) plotted against $J/M_{irr}^2$ vs $m\Omega_0$.]
Corotation: Flat-KV Spin; \( \frac{S}{M_{irr}^2} \) vs \( m\Omega_0 \)
Corotation: Flat-KV Spin Error; \( \frac{(S - S_{Kerr})}{S_{Kerr}} \) vs \( m\Omega_0 \)
Irrotation: Flat-KV Spin; $\frac{S}{M^2} vs m\Omega_0$
Irrotation: Flat-KV Spin Error; $S/S_{Kerr}$ vs $m\Omega_0$
Searching for Killing Vectors

• In general, a 2-surface will not have any Killing vectors.

• We searched for the any Killing vectors or the closest approximation there is to one —
  – Use the “Killing transport” equations\(^4\):

\[
\begin{align*}
v^a D_a \xi_b &= v^a L_{ab} \\
v^a D_a L_{bc} &= 2 R_{cba}^d \xi_d v^a
\end{align*}
\]
Searching for Killing Vectors

- When we have “orbital plane reflection symmetry” we find a true Killing field on the surface!

- Can use this instead of the Conformal Killing vector in the computation of the quasilocal spin.
Killing Vectors on BH Surface
Killing Vectors on BH Surface

CO-Robin-8

Behind
Killing Vector Differences on BH Surface
Killing Vector Differences on BH Surface

CO-Robin-8

Behind
Killing Vector Differences on BH Surface

CO-Robin-8

Axis
Killing Vector Differences on BH Surface

CO-Robin-8

Behind
Killing Vector Differences on BH Surface
Corotation: Flat-KV Spin; \( \frac{S}{M_{irr}^2} \) vs \( m\Omega_0 \)
Corotation: Flat-KV Spin Error; \( \frac{(S - S_{Kerr})}{S_{Kerr}} \) vs \( m\Omega_0 \)
Killing Vector Differences on BH Surface
Killing Vector Differences on BH Surface
Killing Vector Differences on BH Surface
Killing Vector Differences on BH Surface

Ir-Robin-8

Behind

0.0200
0.0175
0.0150
0.0125
0.0100
0.0075
0.0050
0.0025
Killing Vector Differences on BH Surface

Ir-Robin-8

Top

0.0200
0.0175
0.0150
0.0125
0.0100
0.0075
0.0050
0.0025
Irrotation: Flat-KV Spin; \( S/M_{\text{irr}}^2 \) vs \( m\Omega_0 \)
Irrotation: Flat-KV Spin Error; \( S/S_{Kerr} \) vs \( m\Omega_0 \)
True Irrotation

* The spin of the “Irrotational” blacks seem much too large.

– Choice of using $\Omega_0$ in boundary condition for irrotational black holes is probably wrong.
– Choose magnitude of $\beta_i$ to make spin vanish.
Killing Vector Differences on BH Surface

TI-Robin-8
Killing Vector Differences on BH Surface

TI-Robin-8

Behind

1.755
1.750
1.745
1.740
1.735
1.730
Killing Vector Differences on BH Surface
Killing Vector Differences on BH Surface

TI-Robin-8

Behind
Killing Vector Differences on BH Surface
True Irrotation: True-KV Spin; $S/M^2_{\text{irr}}$ vs $m\Omega_0$
True Irrotation: True-KV Spin Error; \( S/S_{Kerr} \) vs \( m\Omega_0 \)
True Irrotation; Maximal Slice; Comparison; $E_b/\mu$ vs $J/\mu m$
Maximal Slice; Comparison of ISCO; $\frac{E_b}{M_{irr}}$ vs $\Omega M_{irr}$

![Graph showing the comparison of ISCO with $\frac{E_b}{M_{irr}}$ vs $\Omega M_{irr}$](image)

**Legend:**
- **CO:** QE
- **CO:** HKV-GGB
- **CO:** PN EOB
- **CO:** PN standard
- **IR:** QE
- **IR:** IVP conf
- **IR:** PN EOB
- **IR:** PN standard
- **TI:** QE

- **Graph Details:**
  - $E_b$ vs $m\Omega_0$
  - Data points and lines representing different models and configurations.
Maximal Slice; Comparison of ISCO; \( \frac{E_b}{M_{irr}} \) vs \( \frac{J}{M_{irr}^2} \)
Maximal Slice; Comparison of ISCO; $J/M_{irr}^2$ vs $ΩM_{irr}$

![Graph showing comparison of ISCO]
References


