Value's Law, Value's Metric

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Abstract

It is argued that the metric space of exchanging commodities is non-euclidean and characteristic of a system governed by a conservation law. The possible candidates for what is conserved in commodity exchange are reviewed with reference to inverted input/output matrices of the British economy. Strong evidence is presented that the conserved substance is labour. The arguments of Mirowski and others regarding the appropriateness of such 'physicalist' arguments are discussed.

1 What is meant by the law of value?

The phrase 'law of value' is little used by Marx, but popular among his followers. It has no precise definition of the type that one would expect for a scientific law. Laws such as Hooke's law or Boyle's law have a concise definition that any chemist or physicist could repeat, but it is doubtful if anywhere in the Marxist literature there exists a comparable definition of the law of value.

On the basis of what Ricardo and Marx wrote on the theory, we would advance the following as a reasonable definition:

The law of value states that value, understood as the labour time socially necessary to produce a commodity, is conserved in the exchange of commodities.

The advantages of this definition are that it is cast in the normal form of a scientific law, it is empirically testable, it has a precise meaning, and it

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emphasizes the fundamental Marxian proposition that value cannot arise in circulation.

In order to justify this formulation of the law we will first take a new look at what Marx (1976, Chapter 1, Section 3) called the 'value-form', and then review the growing body of empirical evidence that justifies the law.

2 Metric spaces

Instead of arguing about the value-form, or exchange-value, in Hegelian terms[1] we will use geometric concepts. This approach, we believe, enables us to pose the problem of exchange-value with greater generality, and at the same time greater concision. It will be necessary to begin with a few definitions.

A *metric space* (S, d) is a space S together with a real-valued function $d : S \otimes S \rightarrow \Re$, which measures the distance between pairs of points $\mathbf{p}, \mathbf{q} \in S$, where d obeys the following axioms:

1. Commutation:
$$d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p}) \, .$$

2. Positivity:
$$0 < d(\mathbf{p}, \mathbf{q}) < \infty \text{ if } \mathbf{p} \neq \mathbf{q} \, .$$

3. Self-identity:
$$d(\mathbf{p}, \mathbf{p}) = 0 \, .$$

4. Triangle inequality:
$$d(\mathbf{p}, \mathbf{q}) \leq d(\mathbf{p}, \mathbf{r}) + d(\mathbf{r}, \mathbf{q}) \, .$$

Examples of metric spaces

Euclidean 2-space. This is the familiar space of planar geometry. If $\mathbf{p}$ and $\mathbf{q}$ are two points with coordinates (p_1, p_2) and (q_1, q_2) respectively, then the distance between these points is given by the pythagorean metric

$$d = \sqrt{\Delta_1^2 + \Delta_2^2} \, ,$$

[1] In the postface to the second edition of *Capital*, Marx (1976, p. 103) noted that he had "coquetted with the mode of expression peculiar to" Hegel in the chapter on the theory of value.

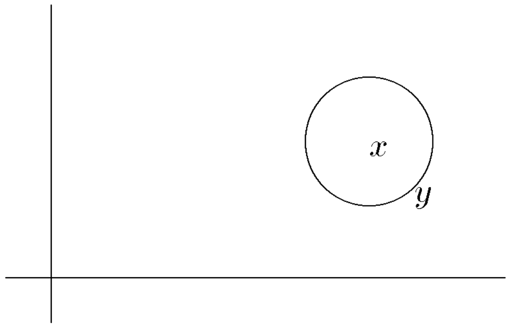

Figure 1: Equality set in Euclidean space

where $\Delta_i = p_i - q_i, i = 1, 2$. It extends to multidimensional vector spaces as

$$d = \sqrt{\Delta_1^2 + \Delta_2^2 + \cdots + \Delta_n^2}\,.$$

Manhattan space. So-called after the Manhattan street plan,[2] the metric is simply the sum of the absolute distances in the two dimensions:

$$d = |\Delta_1| + |\Delta_2|.$$

Equality operations in metric spaces

Let us define two points $\mathbf{q}, \mathbf{r} \in S$ to be equal with respect to $\mathbf{p}$ if they are equidistant from $\mathbf{p}$ under the metric d. Formally,

$$\mathbf{q} =_p \mathbf{r} \text{ if } d(\mathbf{p}, \mathbf{r}) = d(\mathbf{p}, \mathbf{q})\,.$$

Given an equality operator E and a member $\mathbf{q}$ of a set S, we can define an equality subset, that is to say the set whose members are all equal to $\mathbf{q}$ under E. The equality set of $\mathbf{q}$ under $=_p$ using the Euclidean 2-space metric is shown in Figure 1, while Figure 2 shows the corresponding equality set under the Manhattan metric.

3 Commodity bundle space

What, it may be asked, has all this to do with value? Well, value is a metric on commodities. To apply the previous concepts we define *commodity bundle space* as follows. A commodity bundle space of order 2 is the set of

[2] This is also known as a Minkowski metric.

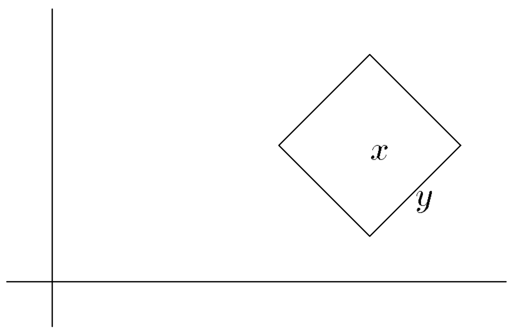

Figure 2: Equality set in Manhattan space

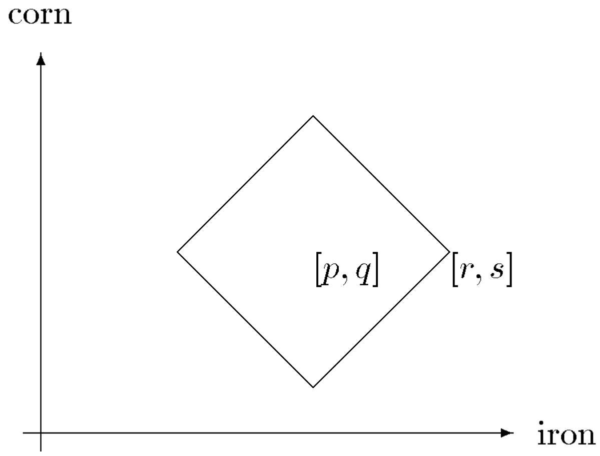

Figure 3: Points equidistant with (e iron, f corn) from (a iron, b corn) in Manhattan space

pairs (ax, by) whose elements are a units of commodity x and b units of commodity y. A commodity bundle space of order 3 is the set of triples (ax, by, cz) whose elements are bundles of a units of x, b units of y, c units of z ... and so on.

Consider for example the commodity bundle space of order 2 composed of bundles of iron and corn. The set of all points equidistant with (e iron, f corn) from (a iron, b corn) under the Manhattan metric is shown in Figure 3.

We have a distinct equality operator, $=_p$, for each point $\mathbf{p} = (p_1$ iron, p_2 corn) in our corn–iron space. Let us consider one particular equality operator, that which defines the equality set of points equidistant from the origin, $=_{(0,0)}$. Whichever metric we take, so long as we use it consistently each point in the space belongs to only one such equality set under the given

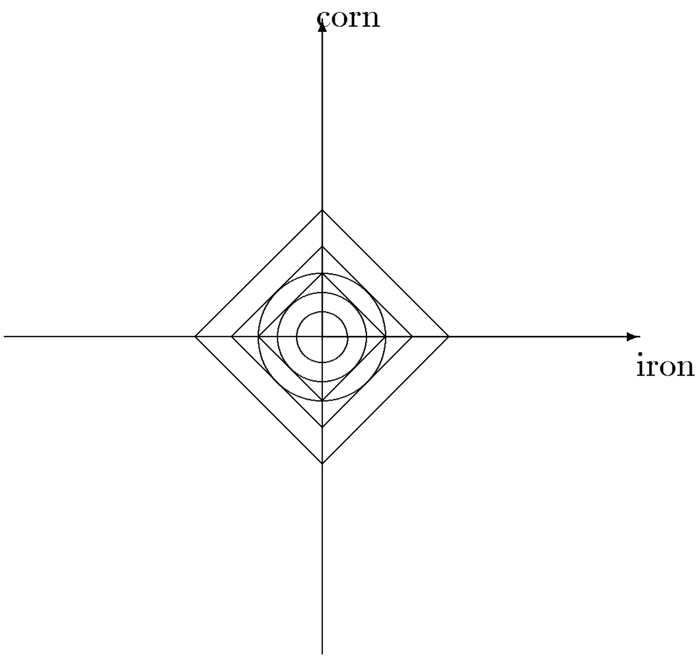

Figure 4: The ordering of equality sets under possible metrics

metric. These equality sets form an ordered set of sets of the space. It follows that any of the metrics could serve as a system of valuation, conceived as a partial ordering imposed upon all bundles. This is shown in Figure 4. Both the diamonds and the conventional circles are, in the relevant space, circles: the diamonds are circles in Minkowski or Manhattan space.

We now advance the hypothesis that if the elements of a set of commodity bundles are mutually exchangeable—that is, if they exchange as equivalents—then they form an equality set under some metric. If this is valid, then by examining the observed equality sets of commodity bundles we can deduce the properties of the underlying metric space.

The metric of commodity bundle space

What is the metric of commodity bundle space? The observed sets of exchangeable bundles constitute the isovalent contours, or *isovals*, in commodity bundle space. We find, in practice, that they are straight lines—known to economists as budget lines (see Figure 5). Note that these extend beyond the axes. Why, we may ask, are they not circles centered on the origin? Commodity space clearly has a non-Euclidean, and for what it is worth, a non-Manhattan geometry, but why? Before attempting an answer to this question it will be useful to make some preliminary points.

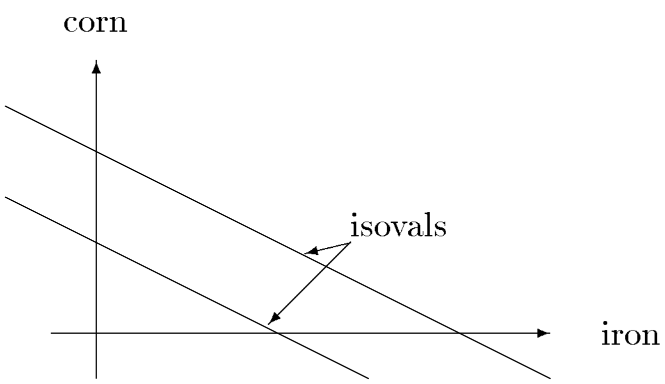

Figure 5: Observed form of the isovals in commodity bundle space

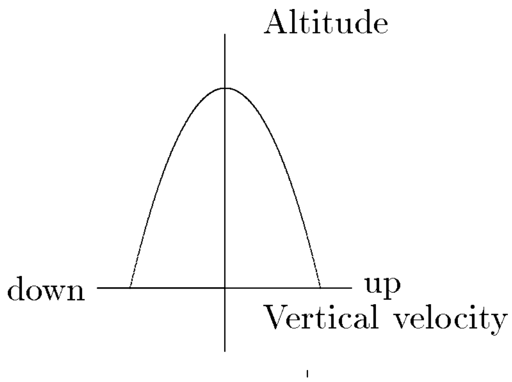

Figure 6: Points in phase space traversed by a projectile thrown upward in a gravitational field

We will call commodity bundle spaces obeying the observed metric of exchange-value, as displayed in the economist's budget lines, *commodity value space*, whereas a commodity bundle space obeying a Euclidean metric we will call *commodity vector space*. (Although our examples have applied to spaces of order two, the argument can be extended to arbitrary hyperspaces.) There is something very particular about the metric of commodity value space, namely $d = |\alpha\Delta_x + \beta\Delta_y|$ where α and β are constants. This metric occurs elsewhere—for instance, in energy conservation.

Consider Figure 6, the graph of position versus velocity for a body thrown up and then falling. All points on the trajectory are 'freely exchangeable' with one another in the course of the time-evolution of the

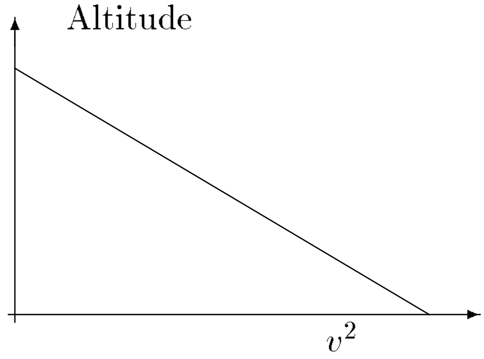

Figure 7: Points in the space of (altitude, velocity squared) traversed by the particle shown in previous figure

system. They may therefore be treated as an equivalence set. This does not look like the equivalence set of commodity value space until we square the velocity axis. This yields the diagram in Figure 7, which looks very much like the budget line in Figure 5. By squaring the velocity axis we obtain a measure proportional to what the physicists term kinetic energy. But this kinetic energy is only revealed through its exchange relation with height. Physics posits a one-dimensional 'substance', energy, whose conservative exchange between different forms underlies the phenomena.

Conjugate isovals

Looking more closely at the metric we have deduced for commodity value space, we can see that our representation of the equality sets as budget lines is only half the story.

Let $\alpha = 1$ and $\beta = 2$ in the metric $d = |\alpha\Delta_x + \beta\Delta_y|$. Taking the point $Q = (2, 1)$ in Figure 8, we can show its equality set with respect to the origin as the line PQR along with its extension in either direction. All such points are at distance 4 from the origin. But by the definition of the metric, the point $Q' = (-2, -1)$ is also at distance 4 from the origin. There thus exists a second equality set on the line $P'Q'R'$ on the opposite side of the origin. In general for a commodity bundle space of order n there will be a conjugate pair of isovals forming parallel hyperplanes of dimension $n - 1$ in commodity vector space.

If the positive isoval corresponds to having positive net wealth, its conjugate corresponds to being in debt to the same amount. There is an obvious echo of this in the practice of double-entry bookkeeping, the effect of which is to ensure that for every credit entry there exists a conjugate debt entry.

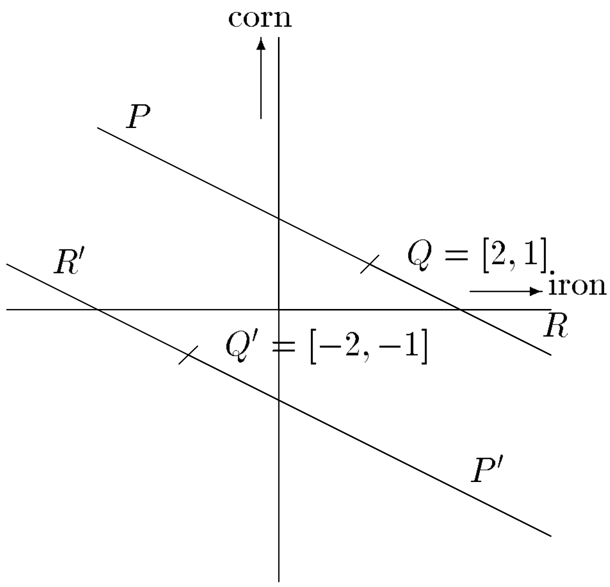

Figure 8: Conjugate pairs of isovals.

Points on an isoval and its conjugate are equidistant from the origin, but not exchangeable with one another. If I have a credit of 1 dollar, I will not readily exchange it for a debt of 1 dollar. This is reflected in the fact that points on an isoval may not be continuously deformed to a point on its conjugate isoval, whereas they may be continuously deformed within the isoval. In other words the isovalent set is topologically disconnected.

Contrast this with what occurs on a Euclidean metric. The points $Q = (2, 1)$ and $Q' = (-2, -1)$ lie on a circle of radius $\sqrt{5}$, along which we may uninterruptedly move from one to the other. The disconnected character of the isovalent set in commodity value space becomes understandable once we realize that this space is a projection of a one dimensional space into an n dimensional one. As such, its unit circles comprise disjoint planes corresponding to the two disjoint points of the unit circle in one-space. It is this characteristic, of being multidimensional projections of one-space, that marks conservative systems.

Implications for value theory

If value were just a matter of providing an ordering or ranking of combinations of goods, then a Euclidean, or indeed any other, metric would pass muster. It is some additional property of the system of commodity production that imposes this specific metric characteristic of a system governed by a conservation law. This fits in rather nicely with the labour theory of value, where social labour would be the embodied substance conserved during exchange relations, which in turn provides us with some justification for casting the law of value in the form of a classical conservation law.

So far, however, this is merely a formal argument: the form of the phenomena is not *inconsistent* with a conservation relation. To justify our formulation we must (1) explain *why* the phenomena are such as to conform to a linear conservation law; (2) show *that* such a law holds empirically; and (3) rule out other potential 'value substances' as alternatives to labour.

4 Why commodity space is non-Euclidean

Spatial metrics are so much part of our mode of thought that to imagine a different metric is conceptually difficult. Most of us have difficulty imagining the curved space–time described by relativity theory, Euclidean metrics being so ingrained in our minds. Conversely, when looking at commodities, a non-Euclidean metric is so ingrained that we have difficulty imagining a Euclidean commodity space.

But it is worth the effort of trying to imagine a Euclidean commodity space, what we referred to earlier as commodity vector space. By bringing to light the implicit contradictions of this idea, we get a better idea of the underlying reasons why value takes the particular form that it does.

Is a Euclidean metric for commodity space internally consistent? In commodity bundle space of order 2 the Euclidean isovals take the form of circles centered on the origin. In higher-order spaces, they take the form of spheres or hyperspheres. (We assume in all cases that some linear scaling of the axes is permitted to convert them into a common set of units.) Let us suppose that the economic meaning of these isovals is that given any pair of points $\mathbf{p}$, $\mathbf{q}$ on an isoval, the bundle of commodities represented by $\mathbf{p}$ will be exchangeable as an equivalent with the bundle represented by $\mathbf{q}$.

If the state of an economic agent is described by his position in this commodity bundle space, then the set of permissible moves that can be made via equivalent exchanges is characterized by unitary operators on commodity vector space. The set of equivalent exchanges of $\mathbf{p}$ is $\{|\mathbf{p}|\,\mathbf{u}$ such that $|\mathbf{u}| = 1\}$ i.e., the radius-preserving rotations of $\mathbf{p}$. Mathematically, this is certainly a consistent system.[3]

But economically, such a system would break down. It says that I can

[3] A very similar model is used in one of the standard formulations of quantum theory to describe possible state transformations (von Neumann, 1955).

exchange one, appropriately defined, unit of corn for one unit of iron, or for any equivalent combination such as ($\frac{1}{\sqrt{2}}$ iron, $\frac{1}{\sqrt{2}}$ corn). But then what is to stop me carrying out the following procedure?

1. Exchange my initial 1 unit of corn for $\frac{1}{\sqrt{2}}$ iron plus $\frac{1}{\sqrt{2}}$ corn.
2. Now sell my $\frac{1}{\sqrt{2}}$ iron for corn, giving me $\frac{1}{\sqrt{2}}$corn.
3. Add my two bundles of corn together, to give a total of $\frac{2}{\sqrt{2}} = \sqrt{2}$ of corn in total.

I end up with more corn than I had at the start, so this cannot be a set of equivalent exchanges. Within the context of the Euclidean metric the second step is illegal, since it involves operating upon one of the coordinates independently. But in the real world, commodities are physically separable, allowing one component of a commodity bundle to be exchanged without reference to others. It is this physical separability of the commodities that makes the observed metric the only consistent one.

The existence of a commodity-producing society, in which the individual components of the wealth held by economic agents can be independently traded, selects out of the possible value metrics one consistent with the law of value. In a hypothetical society in which commodity bundles could not be separated into distinct components, and exchange obeyed a Euclidean metric, the labour theory of value could not hold. However there are several possible conservative value systems consistent with the observed metric. That which is conserved in exchange might be something other than socially necessary labour time.

5 Evidence for labour value conservation

Following the work of Shaikh (1984), there is now a considerable body of econometric evidence in favour of the proposition that relative prices and relative labour values are highly correlated, or in other words, in favour of the law of value as defined above (Petrovic, 1987; Ochoa, 1989; Valle Baeza, 1994; Cockshott, Cottrell and Michaelson, 1995). The general procedure in these studies has been to use data from national input–output tables to calculate the total labour content of the output of each industrial sector, and then to run a cross-sectional regression of the aggregate money price of output, sector by sector, on total labour content. Shaikh (1984) explains

Table 1: Price regressions, UK 1984

	(1)	(2)	(3)	(4)
constant	−0.055 (−2.04)	−0.034 (−1.79)	−0.046 (−2.00)	−0.049 (−2.88)
labour value	1.024 (46.55)	1.014 (63.38)	1.024 (51.20)	
pr. of prod.				1.024 (68.27)
N	101	100	100	100
R^2	.955	.976	.964	.980

Figures in parentheses are t-ratios. All variables in logarithmic form. *Data source*: Central Statistical Office (1988).

the details of the process, and also offers a theoretical argument in favour of a logarithmic specification of the price–value regressions. These studies—utilizing data from the United States, Italy, Yugoslavia, Mexico and the UK—have produced remarkably consistent results, with R^2s of well over .90. It is also noteworthy that there is very little difference, in predictive power over prices, between labour values and 'prices of production'.

Our own findings, from UK data, are presented for reference in Table 1.[4] In the published input–output tables, the labour input is expressed in £. Equation (1) uses labour-value figures calculated on the assumption of a dummy wage-rate of £1 per hour for all industries. This is equivalent to assuming that any wage differentials across industries reflect differential rates of value-creation per clock hour. Equation (2) is the same as (1) except for the exclusion of the oil industry, which is an outlier in the price–value regressions, presumably due to the high rent component (in the Ricardian sense) in oil extraction. Equation (3) (which again excludes the oil industry) uses labour-value figures calculated using wages data from the *New Earnings Survey* to convert backwards from wages to hours for each industry—a correction relative to equation (1) if (and only if) inter-industry wage differentials are the product of extraneous factors, and do not reflect differential rates of value-creation. Finally, equation (4) substitutes prices of produc-

[4] For further details regarding these estimates, see Cockshott, Cottrell and Michaelson .

tion, calculated via a recursive procedure, for labour values (again, excluding oil).

As can be seen from the equation (2) estimates, 'simple' labour values produce an R^2 of nearly 98 percent when the oil sector is excluded and the dummy uniform wage is adopted. Prices of production improve on this performance only marginally.[5]

6 Value: substance versus field

It will not have escaped the reader's notice that there is a 'physicalist' flavour to our argument. Mirowski (1989) has recently had a good deal of innocent fun with the propensity of economists to emulate the queen of the sciences. His critique is directed mainly against the utility theorists, but he does devote some attention to Marx, accusing him of vacillating between a field and a substance theory of value, and, in the context of the transformation problem, of having 'one conservation principle too many'.

The last accusation is valid, but contrary to the Sraffians we believe that on both empirical and theoretical grounds (see Farjoun and Machover, 1983) it is the equalization of the rate of profit that must go. Mirowski's accusation with regard to the contradiction between field and substance theories is relevant to our formulation, however, since it may appear that we have used a substance definition of value in our theoretical discussion and then a field theory for our empirical test. We will attempt to show that the distinction between field and substance theories is more subtle than Mirowski suggests, and that the empirical tests in the literature are not invalidated by this distinction.

By the field version of value theory, Mirowski means the definition of value as current socially necessary, as opposed to historically embodied, labour. He takes as his formal model the now-standard mathematical account of the determination of labour values, as offered by authors such as Morishima (1973) and Steedman (1977). But it is a little unfair to project these twentieth-century formulations, based upon the mathematics of input–output tables, back onto Marx. Marx gave no precise mathematical formulation of the concept of socially necessary labour. The standard modern

[5] It should be noted that due to data limitations our 'prices of production' are calculated on a flow basis—they are prices consistent with the equalization of the rate of profit on the total flow outlay on current inputs. Ochoa (1989) has calculated prices of production on a stock basis for the USA, and finds them to be slighly *less* well correlated with actual prices than are simple labour values.

formulation is just one among several possible definitions of socially necessary labour, and it involves some very unrealistic assumptions. If these assumptions are dropped, and the model made more realistic, the distinction between field and substance theories vanishes.

The standard method of deriving labour values from the linear input–output equations is based upon the assumption that production takes place instantaneously. Marx did not assume this; indeed, he devoted much of Volume II of *Capital* to analysing the turnover times of capital. Any process of determination of prices must operate in time through actual production processes. It is 'socially necessary' that the steel used in the keel of a ship completed today was produced a year or two earlier.[6] The socially necessary labour in steel produced a year ago may differ from that which goes into steel today, but only the former can affect the value of the ship. No real process allows instantaneous information transfer, and market economies are no exception to this rule. If one were to look for a physical analogy, applying the Morishima equations under technological change would be like trying to solve an electrodynamic problem with electrostatics.

The value of the ship will be affected by the value of steel when it was purchased (assuming it was not purchased unnecessarily early). It will also be affected indirectly by the value of steel at a still earlier period, when steel was purchased to make the tools used to build the ship. Generalizing, v_j, the value of commodity j, is affected by v_i, the value of commodity i, at a series of past dates. These effects will be mediated by coupling coefficients $k_1, k_2, \ldots$ corresponding to the fraction of v_j that is made up of the v_i's at times $t-1, t-2, \ldots$. Thus if by v_{ji} we mean the component of v_j that is due to the input of commodity i, both directly and indirectly, we have a difference equation of the form

$$v_{ji,t} = k_1 v_{i,t-1} + k_2 v_{i,t-2} + \cdots .$$

Expressed in continuous terms, this corresponds to a differential equation of the form

$$\frac{dv_j}{dv_i} = \kappa_1 v_i - \kappa_2 \frac{dv_i}{dt} + \kappa_3 \frac{d^2 v_i}{dt^2} - \cdots ,$$

which gives us a highly non-linear field theory. There is no contradiction between this and the substance theory.

[6] We owe this point to Alan Freeman.

Are we then justified in using what are basically the Morishima equations—that is, the wrong field equations—in our verification of a substance theory of value? Yes, because for the constants, k, we have $1 \gg k_1 \gg k_2 \gg k_3 \ldots$ and similarly, $\kappa_2, \kappa_3, \ldots$ will all be very much smaller than κ_1. One can get a feel for their likely scale by examining the Leontief inverse of the UK input–output table. Even when we take two highly cross-linked industries like steel and shipbuilding, we find $0.07 > \sum_i k_i$. We are thus entitled to assume that although there will be some errors in our estimation of values due to using linear field equations, these will be small relative to noise. As a conservation law, the law of value is stochastic and obviously does not hold to the same precision as natural conservation laws (though it should be noted that on an appropriate scale these too are stochastic, due to quantum effects).

7 Conclusion

We have argued that several different metrics for the 'valuation' of bundles of commodities are possible in principle, most of them logically incompatible with the idea that any scalar quantity is conserved in exchange. But the fact that individual commodities are separable, and separately tradable, imposes one particular metric, corresponding to what we called commodity value space—and this metric is consistent with a conservation law. This formal argument does not in itself prove that any identifiable 'substance' is in fact conserved, nor does it establish the credentials of labour time as prime candidate for conservation. That is an empirical matter; and we have shown that the conservation of socially necessary labour time holds as a fairly close approximation in fact. The Ricardian–Marxian 'law of value' may be given a precise definition, on which, moreover, it turns out to be valid.

References

Central Statistical Office, 1988, *Input-Output Tables for the United Kingdom 1984*, London: HMSO.

Cockshott, W. P., Cottrell, A. and Michaelson, G., 1995, 'Testing Marx: some new results from UK data', *Capital and Class*.

Farjoun, E. and Machover, M., 1983, *Laws of Chaos*, London: Verso.

Marx, K., 1967, *Capital*, Volume II, Moscow: Progress Publishers.

Marx, K., 1976, *Capital*, Volume I, Harmondsworth: Penguin/New Left Review.
Mirowski, P., 1989, *More Heat than Light*, Cambridge: Cambridge University Press.
Morishima, M., 1973, *Marx's Economics*, Cambridge: Cambridge University Press.
Ochoa, E. M., 1989, 'Values, prices, and wage–profit curves in the US economy', *Cambridge Journal of Economics*, vol. 13, no. 3, September, pp. 413–29.
Petrovic, P., 1987, 'The deviation of production prices from labour values: some methodology and empirical evidence', *Cambridge Journal of Economics*, vol. 11, no. 3, September, pp. 197–210.
Rodbertus, K., 1904, *Das Kapital*, Griard and Brière.
Roemer, J., 1986, 'Should Marxists be interested in exploitation?' in *Analytical Marxism*, Cambridge: Cambridge University Press.
Shaikh, A., 1984, 'The transformation from Marx to Sraffa', in A. Freeman and E. Mandel (eds) *Ricardo, Marx, Sraffa*, London: Verso, pp. 43–84.
Steedman, I., 1977, *Marx After Sraffa*, London: New Left Books.
Valle Baeza, A., 1994, 'Correspondence between labor values and prices: a new approach', *Review of Radical Political Economics*, vol. 26, no. 2, pp. 57–66.
von Neumann, J., 1955, *Mathematical Foundations of Quantum Mechanics*, Princeton, N.J.: Princeton University Press.