Physics 310/610 – Cosmology

Solution Set J

1. **[10] In class, we demonstrated that for flat rotation curves, $V^2(R) = V_0^2$, the rotational and epicycle angular velocity are related by $\kappa = \Omega\sqrt{2}$. Consider instead the following two cases:**
   (a) **[5] What is the formula for the rotation velocity around a point source of gravity of mass *M*? Find, in this case, a simple ratio between κ and Ω, no more complicated than the one we found for flat rotation curves.**

We start with the equation for κ, namely

$$\kappa^2 = \frac{2V_0^2}{R_0^2} + \frac{1}{R_0}\frac{d}{dR}V^2(R)\Big|_{R_0}$$

As calculated in class, the formula for the velocity of circular orbits around a mass *M* is $V^2(R) = GM/R$. We therefore have

$$\frac{d}{dR}V^2(R)\Big|_{R=R_0} = -\frac{GM}{R_0^2}$$

Substituting in, we have

$$\kappa^2 = \frac{2V_0^2}{R_0^2} - \frac{1}{R_0}\frac{GM}{R_0^2} = \frac{2V_0^2}{R_0^2} - \frac{V_0^2}{R_0^2} = \frac{V_0^2}{R_0^2} = \Omega^2,$$

or, to simplify, $\kappa = \Omega$. It can't get any simpler than that!

   (b) **[5] What is the formula for the rotation velocity inside a sphere with uniform density ρ? Find, in this case, a simple ratio between κ and Ω, no more complicated than the one we found for flat rotation curves.**

We again use the formula $V^2(R) = GM/R$, but we replace *M* by the mass enclosed within a sphere or radius *R*, which is $M = \frac{4}{3}\pi R^3 \rho$. We therefore have $V^2(R) = \frac{4}{3}\pi\rho R^2$, and hence

$$\frac{d}{dR}V^2(R)\Big|_{R=R_0} = \tfrac{8}{3}\pi G\rho_0 R$$

The frequency for epicycles, therefore, is

$$\kappa^2 = \frac{2V_0^2}{R_0^2} + \frac{1}{R_0}\frac{8\pi}{3}G\rho_0 R_0 = \frac{2V_0^2}{R_0^2} + \frac{8\pi}{3}G\rho_0 = \frac{2V_0^2}{R_0^2} + 2\frac{V_0^2}{R_0^2} = \frac{4V_0^2}{R_0^2} = 4\Omega^2,$$

or, to simplify, $\kappa = 2\Omega$. This is as simple, if not simpler, than the formula for flat rotation curves.

2. [10] For the Milky Way galaxy at the radius of the Sun, find the three frequencies Ω, ν, and κ (in Myr^{-1}) and the corresponding periods (in Myr) in the neighborhood of the Sun. Assume we are 8.18 kpc from the center, that our galaxy has a flat rotation curve with $V_0 = 220\text{ km/s}$ and the local mass density is $\rho = 0.07\ M_\odot \text{pc}^{-3}$. In one orbit of the galaxy, T_ϕ, how many cycles of up and down motion does the Sun undergo?

The orbital frequency is just

$$\Omega = \frac{V_0}{R_0} = \frac{220\text{ km/s}}{8.18\text{ kpc}} \cdot \frac{\text{pc}}{3.086\times10^{16}\text{ m}} \cdot \frac{3.156\times10^{7}\text{ s}}{\text{y}} \cdot \frac{10^3\text{ m}}{\text{km}} \cdot \frac{10^6\text{ y}}{\text{My}} \cdot \frac{\text{kpc}}{10^3\text{ pc}} = 0.0275\text{ My}^{-1}$$

The epicycle frequency is

$$\kappa = \sqrt{2}\Omega = \sqrt{2}\cdot 0.0275\text{ My}^{-1} = 0.0389\text{ My}^{-1}$$

The hard one is the up and down frequency, which is given by

$$\nu = \sqrt{4\pi G\rho_0} = \sqrt{\frac{4\pi\left(6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2\right)\left(0.07\,M_\odot/\text{pc}^3\right)\left(1.989\times10^{30}\text{ kg}/M_\odot\right)}{\left(3.086\times10^{16}\text{ m/pc}\right)^3}}$$

$$= \left(1.99\times10^{-15}\text{ s}^{-1}\right)\cdot\frac{3.156\times10^{7}\text{ s}}{\text{y}}\cdot\frac{10^6\text{ My}}{\text{y}} = 0.0629\text{ My}^{-1}$$

The corresponding periods are

$$T_\phi = \frac{2\pi}{\Omega} = \frac{2\pi}{0.0275\text{ My}^{-1}} = 228\text{ My},$$

$$T_R = \frac{2\pi}{\kappa} = \frac{2\pi}{0.0389\text{ My}^{-1}} = 161\text{ My},$$

$$T_z = \frac{2\pi}{\nu} = \frac{2\pi}{0.0629\text{ My}^{-1}} = 99.9\text{ My}.$$

Obviously, it makes about 2.28 vertical oscillations for each trip around the galaxy.

Note: Problem 3 requires that you do problem 2 first.

3. **[10] Assume the Sun is currently at $z = 0$ and $R = R_0$. The former implies that the Sun is right in the galactic plane (it's pretty close), the latter implies that the Sun is currently moving at exactly the right radial velocity for a circular orbit (it is not, so sue me).**
   (a) [5] If the Sun is currently moving upwards at 8 km/s, determine the maximum distance z_0 (in pc) the Sun will reach above the plane. Also determine how long from now (in Myr) it will reach this position.

Well, the equation describing the vertical position is $z = z_0 \sin(\nu t)$. We could have used cosine instead, but since we are told it is currently at $z = 0$, we need sine, not cosine. The velocity in the z-direction is the derivative of this, or $v_z = z_0 \nu \cos(\nu t)$, which evaluated at $t = 0$ gives $v_z = z_0 \nu$. It follows that

$$z_0 = \frac{v_z}{\nu} = \frac{8\ \text{km/s}}{0.0629\ \text{My}^{-1}} \cdot \frac{10^6\ \text{y}}{\text{My}} \cdot \frac{3.156 \times 10^7\ \text{s}}{\text{y}} \cdot \frac{\text{pc}}{3.086 \times 10^{13}\ \text{km}} = 130\ \text{pc}.$$

The time this will occur is when z is at a maximum, which is when $\sin(\nu t)$ is maximized, which occurs when $\nu t = \frac{1}{2}\pi$, so

$$t_z = \frac{\pi}{2\nu} = \frac{\pi}{2(0.0629\ \text{My}^{-1})} = 25\ \text{My}$$

(b) [5] If the Sun is currently moving inwards at 11 km/s, determine the maximum amount ΔR (in pc) that it will drift in from its current distance of 8.18 kpc. Also determine how long from now (in Myr) it will reach this position.

This time the equation describing the radial position is $R = R_0 + (\Delta R)\sin(\kappa t)$. Once again, we argue that the sine is appropriate, instead of the cosine. The velocity is given by the time derivative, or $v_R = (\Delta R)\kappa \cos(\kappa t)$, so the velocity now is $v_R = (\Delta R)\kappa$. We therefore have

$$\Delta R = \frac{v_R}{\kappa} = \frac{-11\ \text{km/s}}{0.0389\ \text{My}^{-1}} \cdot \frac{10^6\ \text{y}}{\text{My}} \cdot \frac{3.156 \times 10^7\ \text{s}}{\text{y}} \cdot \frac{\text{pc}}{3.086 \times 10^{13}\ \text{km}} = -289\ \text{pc}$$

The minus sign just means that it is inward, not outwards. It will achieve this distance once again when $\sin(\kappa t)$ is at a maximum, which is when $\kappa t = \frac{1}{2}\pi$, so that

$$t_R = \frac{\pi}{2\kappa} = \frac{\pi}{2(0.0389\ \text{My}^{-1})} = 40.4\ \text{My}.$$

Physics 610: Only do this problem if you are in the graduate version of this course

4. [20] For vertical motion, we assumed the density was of the form $\rho = \rho_0$. A more realistic expression would be $\rho = \rho_0 e^{-|z|/h_z}$.

(a) [5] For this density, find an expression for the scalar potential $\Phi(z)$ as a function of *z*.

Since the mass distribution is symmetric around $z = 0$, it is reasonable to assume the potential is symmetric about it as well, so that $\Phi(z) = \Phi(-z)$. We therefore will assume that we are working at $z > 0$. We first find the gravitational acceleration:

$$\mathbf{g}(Z) = -4\pi G\hat{\mathbf{z}}\int_0^Z \rho(z)\,dz = -4\pi G\rho_0\hat{\mathbf{z}}\int_0^Z e^{-|z|/h_z}\,dz = -4\pi G\rho_0\hat{\mathbf{z}}\int_0^Z e^{-z/h_z}\,dz = 4\pi G\rho_0 h_z\, e^{-z/h_z}\Big|_0^Z\,\hat{\mathbf{z}}$$
$$= 4\pi G\rho_0 h_z\left(e^{-Z/h_z} - 1\right)\hat{\mathbf{z}}$$

We then find the gravitational potential using

$$\Phi(z) = -\int \mathbf{g}\cdot d\mathbf{s} = -4\pi G\rho_0 h_z\int\left(e^{-z/h_z} - 1\right)dz = 4\pi G\rho_0 h_z\left(z + h_z e^{-z/h_z} - h_z\right).$$

The constant of integration by demanding $\Phi(0) = 0$. We now simply generalize to include the case $z < 0$, so we have

$$\Phi(z) = 4\pi G\rho_0 h_z\left(|z| + h_z e^{-|z|/h_z} - h_z\right).$$

(b) [5] Suppose a star is bobbing up and down, reaching maximum height $z = \pm h$. Using conservation of energy, find the velocity *v* as a function of its height *z* and maximum height *h*.

The potential energy is $m\Phi(z)$; the kinetic energy is $\frac{1}{2}mv^2$. The total energy, which is conserved, is then $E = m\Phi(z) + \frac{1}{2}mv^2$. When the star is at its maximum/minimum position $z = \pm h$, the kinetic energy must be zero, so $E = m\Phi(\pm h)$. Hence $m\Phi(\pm h) = m\Phi(z) + \frac{1}{2}mv^2$. Cancelling the *m*'s and solving for v^2, we find

$$v^2 = 2\Phi(\pm h) - 2\Phi(z) = 2\cdot 4\pi G\rho_0 h_z\left(h + h_z e^{-h/h_z} - h_z - |z| - h_z e^{-|z|/h_z} + h_z\right),$$
$$v = \frac{dz}{dt} = \sqrt{8\pi G\rho_0 h_z\left(h + h_z e^{-h/h_z} - |z| - h_z e^{-|z|/h_z}\right)}.$$

(c) [5] Find an integral form for the time it takes to go through one-fourth of a cycle, from z = 0 to h, then quadruple it to get the total period T, as a function of h. The corresponding formula for uniform density would be $T = \dfrac{2\pi}{\sqrt{4\pi G\rho_0}}$.

The time for it to go from 0 to h would be

$$T_{1/4} = \int_0^{T_{1/4}} dt = \int_0^h dz \frac{dt}{dz} = \int_0^h \frac{dz}{v} = \int_0^h \frac{dz}{\sqrt{8\pi G\rho_0 h_z \left(h + h_z e^{-h/h_z} - z - h_z e^{-z/h_z}\right)}}.$$

The time to return to $z = 0$ is the same as this, and the time for the other half of the cycle, when z is negative, is exactly the same. In total, therefore,

$$T = \int_0^h \frac{4dz}{\sqrt{8\pi G\rho_0 \left(h + h_z e^{-h/h_z} - z - h_z e^{-z/h_z}\right)}}.$$

To make the formulas look as similar as possible, define x by $z = h_z x$. Then the integral becomes

$$T = \int_0^{h/h_z} \frac{4h_z dx}{\sqrt{8\pi G\rho_0 \left(h + h_z e^{-h/h_z} - h_z x - h_z e^{-x}\right)}} = \frac{1}{\sqrt{\pi G\rho_0}} \int_0^{h/h_z} \frac{\sqrt{2}dx}{\sqrt{h/h_z + e^{-h/h_z} - x - e^{-x}}}$$

This makes the formula as similar as possible. It is clear that all that is left is the integral, which is a function of the single variable h/h_z.

(d) [5] Evaluate the integral in part (c) numerically for $h = \frac{1}{2}h_z$, $h = h_z$, and $h = 2h_z$. Compare to the result for uniform density.

Since it's just a numerical integral, we simply let some sort of program do it for us. I like Maple.

```
> feval:= proc(y) int(sqrt(2/(y+exp(-y)-x-exp(-x))),x=0.0..y)
  end proc
> feval(.5); feval(1.0); feval(2.0);
```

We find

$$T\left(h = \tfrac{1}{2}h_z\right) = \frac{3.4792}{\sqrt{\pi G\rho_0}}, \quad T\left(h = h_z\right) = \frac{3.8227}{\sqrt{\pi G\rho_0}}, \quad T\left(h = 2h_z\right) = \frac{4.5125}{\sqrt{\pi G\rho_0}}.$$

All of these numerators would be π if we assumed the density was uniform.