Physics 310/610 – Cosmology

Solution Set M

1. **[15] The critical density is the density required to have Ω = 1. Assuming Hubble's constant is $H_0 = 67.8\text{ km/s/Mpc}$,**

   (a) [7] Find the critical density. Write your answer in kg/m³ and in $M_\odot/\text{kpc}^3$.

The critical density is given by the formula $H_0^2 = \frac{8}{3}\pi G\rho_c$. Solving for the critical density, we have

$$\rho_c = \frac{3H_0^2}{8\pi G} = \frac{3(67.8\text{ km/s/Mpc})^2}{8\pi\left(6.674\times10^{-11}\text{ m}^3\text{kg}^{-1}\text{s}^{-2}\right)\left(10^6\text{ pc/Mpc}\right)^2\left(3.086\times10^{13}\text{ km/pc}\right)^2}$$
$$= 8.633\times10^{-27}\text{ kg/m}^3.$$

We can write this in $M_\odot/\text{kpc}^3$ using

$$\rho_c = \frac{\left(8.633\times10^{-27}\text{ kg/m}^3\right)\left(3.086\times10^{16}\text{ m/pc}\right)^3\left(10^3\text{ pc/kpc}\right)^3}{\left(1.989\times10^{30}\text{ kg}/M_\odot\right)} = 128\ M_\odot\text{kpc}^{-3}.$$

   (b) [8] The actual value of Ω for ordinary matter is only $\Omega_b = 0.0484$. If this is all in the form of hydrogen atoms, what is the number density of hydrogen atoms per cubic meter?

We multiply $\rho_b = \Omega_b\rho_c$ and find

$$\rho_b = \Omega_b\rho_c = (0.0484)\left(8.633\times10^{-27}\text{ kg/m}^3\right) = 4.178\times10^{-28}\text{ kg/m}^3.$$

The density will be the mass of hydrogen times the number density, $\rho_b = n_H m_H$. The atomic mass of a hydrogen is 1.00794, which you divide by Avogadgro's number, $N_A = 6.022\times10^{23}$ to give the mass in grams, i.e.,

$$m_H = \frac{1.00794\text{ g/mol}}{6.022\times10^{23}\text{ / mol}} = 1.674\times10^{-24}\text{ g} = 1.674\times10^{-27}\text{ kg}.$$

The number density will therefore be

$$n_H = \frac{\rho_b}{m_H} = \frac{4.178\times10^{-28}\text{ kg/m}^3}{1.674\times10^{-27}\text{ kg}} = 0.250\text{ m}^{-3}.$$

So there is about one hydrogen atom per four cubic meters, or more accurately, one proton or neutron per four cubic meters.

2. [10] We have mostly been neglecting the photons. As we will discover shortly, the universe is filled with electromagnetic radiation at a temperature $T_r = 2.725\text{ K}$.

(a) [7] Find the energy density *u*. Also find the mass density $\rho_r = u/c^2$.

The energy density is given by the formula

$$u = \frac{\pi^2 (k_B T)^4}{15(\hbar c)^3} = \frac{\pi^2 \left[(1.3807\times10^{-23}\text{ J/K})(2.725\text{ K})\right]^4}{15\left[(1.0546\times10^{-34}\text{ J}\cdot\text{s})(2.998\times10^{8}\text{ m/s})\right]^3} = 4.17\times10^{-14}\text{ J/m}^3$$

We then divide this by c^2 to get the mass density

$$\rho_r = \frac{u}{c^2} = \frac{4.17\times10^{-14}\text{ J/m}^3}{(2.998\times10^{8}\text{ m/s})^2} = 4.642\times10^{-31}\text{ kg/m}^3$$

(b) [3] What is the contribution Ω_r to the total energy density of the universe?

To get this, we simply divide this by the critical density found in problem 1. So we have

$$\Omega_r = \frac{\rho_r}{\rho_c} = \frac{4.642\times10^{-31}\text{ kg/m}^3}{8.633\times10^{-27}\text{ kg/m}^3} = 5.38\times10^{-5}\,.$$

This is pretty small.

3. [5] In class I claimed that any point on a 3-sphere of radius *a* could be written as

$$x = a\sin\psi\sin\theta\cos\phi,\quad y = a\sin\psi\sin\theta\sin\phi,\quad z = a\sin\psi\cos\theta,\quad w = a\cos\psi\,.$$

Show that these points do, in fact, constitute a 3-sphere of radius *a*.

To show this, we simply check that the sum of the distances from the origin equals a^2. We have

$$\begin{aligned}
s^2 &= x^2 + y^2 + z^2 + w^2 \\
&= a^2\sin^2\psi\sin^2\theta\cos^2\phi + a^2\sin^2\psi\sin^2\theta\sin^2\phi + a^2\sin^2\psi\cos^2\theta + a^2\cos^2\psi \\
&= a^2\sin^2\psi\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + a^2\sin^2\psi\cos^2\theta + a^2\cos^2\psi \\
&= a^2\sin^2\psi\left(\sin^2\theta + \cos^2\theta\right) + a^2\cos^2\psi \\
&= a^2\sin^2\psi + a^2\cos^2\psi \\
&= a^{2\cdot}\,.
\end{aligned}$$

That was pretty easy!

Graduate problem: Only do this problem if you are in PHY 610

4. **[15] A closed universe has space distance formula** $ds^2 = a^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$. **Our goal in this problem is to find the volume of the universe. The *metric* g_{ij} is just the 3×3 matrix defined by** $ds^2 = \sum_i \sum_j g_{ij} dx^i dx^j$ **.**

   (a) **[5] Find the volume of the universe, which is given by** $V = \int \sqrt{\det\left(g_{ij}\right)} d^3x$**. Note the determinant** $\det\left(g_{ij}\right)$ **takes care of any necessary factors in the integral. You may have to think a bit (or ask) about the limits on all the angular variables.**

   First, the metric is given by

$$g_{ij} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2\sin^2\psi & 0 \\ 0 & 0 & a^2\sin^2\psi\sin^2\theta \end{pmatrix}$$

The determinant is then easily seen to be $\det\left(g_{ij}\right) = a^6\sin^4\psi\sin^2\theta$. To figure out the range of integration is a little trickier. The angles θ and ϕ are very similar to those used in ordinary spherical coordinates, in that $0 < \theta < \pi$ and $0 < \phi < 2\pi$. From the formula $w = a\cos\psi$, you can figure out that to cover the whole sphere, with w running from a to $-a$, we are going to have to use $0 < \psi < \pi$. Hence the volume of the universe is going to be

$$V = \int_0^\pi d\psi \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{a^6\sin^4\psi\sin^2\theta} = a^3\int_0^\pi \sin^2\psi\, d\psi \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$
$$= 2\pi a^3\, \tfrac{1}{2}\left[\psi - \tfrac{1}{2}\sin\left(2\psi\right)\right]_0^\pi \left[-\cos\theta\right]_0^\pi = 2\pi a^3\, \tfrac{1}{2}\left[\pi\right]\left[2\right] = 2\pi^2 a^3.$$

   (b) **[5] Using the Friedman equation with $k = +1$ (closed universe), find an expression for a_0 in terms of Ω and H_0.**

   If $\Omega \le 1$, then the universe is either flat or open, and in both cases probably infinite volume, so there is no point in finding the minimum size. But if $\Omega > 1$, then we necessarily have a finite universe. Keeping in mind that $\tfrac{8}{3}\pi G\rho_0 \equiv H_0^2\Omega$, the Friedman equation can easily be rewritten as

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho - \frac{kc^2}{a^2},$$
$$H_0^2 = H_0^2\Omega - \frac{c^2}{a_0^2},$$
$$c^2 a_0^{-2} = H_0^2\left(\Omega - 1\right),$$
$$a_0 = \frac{c}{H_0\sqrt{\Omega - 1}}.$$

(c) [5] Experimentally, H_0 = 67.8 km/s/Mpc, and $\Omega = 1.0023 \pm 0.0055$. Assuming $1 < \Omega < 1.01$, find a minimum size for the universe a_0 in Gpc and a minimum volume in Gpc^3.

The smaller Ω is, the larger the universe is, so to get a minimum size for the universe we must take Ω as large as possible. So we conclude

$$a_0 > \left.\frac{c}{H_0\sqrt{\Omega-1}}\right|_{\Omega=1.01} = \frac{\left(2.998\times10^5 \text{ km/s}\right)}{\left(67.8 \text{ km/s/Mpc}\right)\sqrt{1.01-1}} = 44,200 \text{ Mpc} = 44.2 \text{ Gpc}.$$

The volume, therefore, is

$$V_0 > 2\pi^2 a_{\min}^3 = 2\pi^2\left(44.2 \text{ Gpc}\right)^3 = 1.71\times10^6 \text{ Gpc}^3.$$

In other words, the universe is big.