Physics 310 – Cosmology

Solution Set N

1. **[20] The density of ordinary matter (baryons), dark matter, radiation, and cosmological constant are currently about**

$$\rho_{b0} = 4.196\times10^{-28}\ \text{kg/m}^3,\quad \rho_{r0} = 4.642\times10^{-31}\ \text{kg/m}^3,$$
$$\rho_{d0} = 2.235\times10^{-27}\ \text{kg/m}^3,\quad \rho_{\Lambda0} = 5.966\times10^{-27}\ \text{kg/m}^3.$$

The baryons and dark matter scale as a^{-3}, the radiation as a^{-4}, and the cosmological constant does not scale.

(a) [5] Note that $a^{-1} \propto z+1$. Write a simple formula for the total density as a function of the red-shift *z*.

It is pretty clear that we would have $\rho_b, \rho_d \propto (z+1)^3$ and $\rho_r \propto (z+1)^4$, while ρ_Λ is constant. Since the factors of $z + 1$ are just 1 today, it follows that

$$\rho = (\rho_{b0} + \rho_{d0})(z+1)^3 + \rho_{r0}(z+1)^4 + \rho_{\Lambda0}$$

If we prefer, we can substitute the explicit values from above.

(b) [5] Find the red-shift when the matter ($\rho_m = \rho_b + \rho_d$) matched the cosmological constant, and when the cosmological constant was only 1% of the matter.

We want to have $(\rho_{b0} + \rho_{d0})(z+1)^3 = \rho_{\Lambda0}$. Solving for *z*, we have

$$(z+1)^3 = \frac{\rho_{\Lambda0}}{\rho_{b0} + \rho_{d0}} = \frac{5.966\times10^{-27}\ \text{kg/m}^3}{4.196\times10^{-28}\ \text{kg/m}^3 + 2.235\times10^{-27}\ \text{kg/m}^3} = 2.247,$$
$$z = \sqrt[3]{2.247} - 1 = 0.310.$$

So the red-shift when matter *stopped* dominating the universe was actually very modest. When the cosmological constant was only 1% of the matter, we would have

$$(z+1)^3 = \frac{\rho_{\Lambda0}}{0.01(\rho_{b0} + \rho_{d0})} = 100(2.247) = 224.7,$$
$$z = \sqrt[3]{224.7} - 1 = 5.08.$$

Hence the cosmological constant is relevant at the 1% level only for $z < 5$.

(c) [5] Recombination (to be studied soon) occurred at $z+1 = 1091$. Find the ratio of matter to radiation at this time. Is the cosmological constant important at this time?

We find

$$\frac{\rho_m}{\rho_r} = \frac{(\rho_{b0}+\rho_{d0})(z+1)^3}{\rho_{r0}(z+1)^4} = \frac{4.196\times10^{-28}\text{ kg/m}^3 + 2.235\times10^{-27}\text{ kg/m}^3}{(4.642\times10^{-31}\text{ kg/m}^3)(z+1)} = \frac{5719}{1091} = 5.24.$$

So matter still dominated, but only by a factor of five. In fact, if you include neutrinos, the factor will be even smaller. In contrast, we have

$$\frac{\rho_\Lambda}{\rho_m} = \frac{\rho_{\Lambda 0}}{(\rho_{b0}+\rho_{d0})(z+1)^3} = \frac{5.966\times10^{-27}\text{ kg/m}^3}{(4.196\times10^{-28}\text{ kg/m}^3 + 2.235\times10^{-27}\text{ kg/m}^3)(z+1)^3} = \frac{2.25}{1091^3}$$
$$= 1.73\times10^{-9}.$$

Clearly, the cosmological constant is irrelevant at this time.

(d) [5] Primordial nucleosynthesis occurred around $z = 3.4\times10^8$. Find the density of <u>just the ordinary matter</u> at this red-shift. Compare to the density of air at standard temperature and pressure.

The density of the ordinary matter would be

$$\rho_b = \rho_{b0}(1+z)^3 = (4.196\times10^{-28}\text{ kg/m}^3)(3.4\times10^8)^3 = 0.0165\text{ kg/m}^3.$$

By comparison air at standard temperature and pressure is 1.274 kg/m^3. So the universe was not very dense, only 1.3% of the density of air, even back then. The temperature at the time, however, was $T = T_0(1+z) = 3.4\times10^8\times2.725\text{ K} = 9.26\times10^8\text{ K}$, or nearly a billion Kelvin.

2. [10] The current age of the universe was found in class, assuming radiation is irrelevant, and that $\Omega_m + \Omega_\Lambda = 1$, was given by

$$t_0 = H_0^{-1}\int_0^1 \frac{dx}{\sqrt{\Omega_m/x + \Omega_\Lambda x^2}}$$

(a) [3] Given that *x* is simply $x = a/a_0$, argue that *x* corresponds to a simple function of the red-shift *z*. What must change about the upper limit of integration if we want to know the age of the universe *t* at red-shift *z*? (only the range of integration changes).

We have from our notes that $a/a_0 = (1+z)^{-1}$, so it is obvious that $x = (1+z)^{-1}$. If we want to go from the beginning of time ($z = \infty$), to red-shift *z*, then the integral must be modified to

$$t = H_0^{-1}\int_0^{(1+z)^{-1}} \frac{dx}{\sqrt{\Omega_m/x + \Omega_\Lambda x^2}}$$

(b) [7] Argue that if *z* is large, the term $\Omega_\Lambda x^2$ is irrelevant. Based on this, find a relationship for the age of the universe *t* as a function of red-shift *z*.

As shown in the previous problem, the contribution of the cosmological constant is less than 1% of the contribution of matter for $z > 5$. Hence if $z > 5$, we ignore the $\Omega_\Lambda x^2$ term. We then proceed to perform the integration:

$$t = H_0^{-1}\int_0^{(1+z)^{-1}} \frac{dx}{\sqrt{\Omega_m/x+\Omega_\Lambda x^2}} \approx H_0^{-1}\int_0^{(1+z)^{-1}} \frac{dx}{\sqrt{\Omega_m/x}} = H_0^{-1}\Omega_m^{-1/2}\int_0^{(1+z)^{-1}} x^{1/2}dx$$

$$= H_0^{-1}\Omega_m^{-1/2}\,\tfrac{2}{3}x^{3/2}\Big|_0^{(1+z)^{-1}} = \frac{2}{3H_0\sqrt{\Omega_m}\left(1+z\right)^{3/2}}.$$

Graduate Problem: Only do this problem if you are in PHY 610

3. [15] The age at any stage for the universe, assuming it is composed exclusively of matter and dark matter, is given by an integral of the form

$$t_0 = H_0^{-1}\int \frac{dx}{\sqrt{\Omega_m/x+\Omega_\Lambda x^2}}$$

(a) What would be the limits if we want to know how long it takes from now until the universe is infinite in size? Convince yourself that this will take infinite time.

So, since $x = a/a_0$, we have $x = 1$ now and $x = \infty$ when the universe gets to infinite size. So we have

$$t_\infty = H_0^{-1}\int_1^\infty \frac{dx}{\sqrt{\Omega_m/x+\Omega_\Lambda x^2}}$$

At present, the dark energy dominates the matter, and as x becomes large, we can increasingly ignore Ω_m/x compared to $\Omega_\Lambda x^2$. Hence we can approximate

$$t_\infty \approx H_0^{-1}\int_1^\infty \frac{dx}{\sqrt{\Omega_\Lambda x^2}} = \frac{1}{H_0\sqrt{\Omega_\Lambda}}\int_1^\infty \frac{dx}{x} = \frac{1}{H_0\sqrt{\Omega_\Lambda}}\ln\left(x\right)\Big|_1^\infty = \infty.$$

So it is indeed infinite.

(b) The formula above assumes that the dark matter does not scale, that is, that $\rho_\Lambda \propto a^0$. Suppose instead that $\rho_\Lambda \propto a^n$, with *n* a small positive number. How would the integral change?

The formula was derived starting from the Friedman equation for a flat universe, namely

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho$$

The densities today are given by

$$\tfrac{8}{3}\pi G\rho_{m0} = H_0^2\Omega_m, \quad \tfrac{8}{3}\pi G\rho_{\Lambda 0} = H_0^2\Omega_\Lambda$$

Assuming these scale as a^{-3} and a^n respectively, we would then have, at arbitrary time,

$$\tfrac{8}{3}\pi G\rho_m = H_0^2\Omega_m\left(a/a_0\right)^{-3}, \quad \tfrac{8}{3}\pi G\rho_\Lambda = H_0^2\Omega_\Lambda\left(a/a_0\right)^{n}.$$

Substituting this into the Friedman equation, we have

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[\Omega_m\left(a/a_0\right)^{-3} + \Omega_\Lambda\left(a/a_0\right)^{n}\right].$$

Now, recalling that $x = a/a_0$, this equation can be rewritten as

$$\frac{\dot{x}^2}{x^2} = H_0^2\left[\Omega_m x^{-3} + \Omega_\Lambda x^{n}\right].$$

We then rearrange this as

$$\left(\frac{dx}{dt}\right)^2 = H_0^2\left[\Omega_m x^{-1} + \Omega_\Lambda x^{2+n}\right],$$

$$t = \int dt = \int \frac{dx}{\left(dx/dt\right)} = \frac{1}{H_0}\int \frac{dx}{\sqrt{\Omega_m x^{-1} + \Omega_\Lambda x^{2+n}}}.$$

The limits until the end of the universe from now, as before, will be 1 and ∞, so we have

$$t_\infty = H_0^{-1}\int_1^\infty \frac{dx}{\sqrt{\Omega_m/x + \Omega_\Lambda x^{2+n}}}.$$

(c) Since dark energy currently dominates, and will do so more in the future, ignore Ω_m/x. Perform the integral you found in part (b), assuming *n* is positive.

This makes the integral pretty easy. We have

$$t_\infty \approx \frac{1}{H_0}\int_1^\infty \frac{dx}{\sqrt{\Omega_\Lambda x^{2+n}}} = \frac{1}{H_0\sqrt{\Omega_\Lambda}}\int_1^\infty x^{-1+\frac{1}{2}n}dx = \frac{1}{H_0\sqrt{\Omega_\Lambda}}\frac{1}{\frac{1}{2}n}x^{\frac{1}{2}n}\Big|_1^\infty = \frac{2}{nH_0\sqrt{\Omega_\Lambda}}.$$

(d) We currently have $\Omega_\Lambda = 0.691$ and $H_0^{-1} = 14.4\text{ Gyr}$. Observation suggests $n = -0.06 \pm 0.16$. Assuming $n < 0.10$, what is the minimum time until the end of the universe? This end is called the "Big Rip."

The smaller *n* is, the longer the time. Hence the minimum time would be when

$$t_\infty > \frac{2}{n_{\min}H_0\sqrt{\Omega_\Lambda}} = \frac{2\times 14.4\text{ Gyr}}{0.1\sqrt{0.691}} = 346\text{ Gyr}.$$

Hence the Big Rip, if it occurs at all, will be more than one-third of a trillion years in the future.