Physics 310/610 – Cosmology

Solution Set Q

1. **[10] In this problem we are going to estimate the number of neutrinos in the current universe.**
   (a) For each of the types of neutrino, find the current number density in m^{-3} using the neutrino temperature we found in class.

   This was actually done in class, but we can redo it pretty easily. We have

$$n_{\nu i} = \frac{\zeta(3)}{\pi^2}\left(\frac{k_B T_\nu}{\hbar c}\right)^3 2\cdot\frac{3}{4} = \frac{3\zeta(3)}{2\pi^2}\left[\frac{\left(8.617\times10^{-5}\text{ eV/K}\right)\left(1.945\text{ K}\right)}{1.973\times10^{-7}\text{ eV}\cdot\text{m}}\right]^3 = 1.120\times10^8\text{ m}^{-3}\,.$$

You would then multiply by three to get the total for all three species.

   (b) The neutrinos are known to have a small difference in mass, which we will treat as zero. One of them is known to be lighter than about 2 eV/c^2. Assuming this is the mass of all three neutrinos, find the total mass density (in kg/m^3) for all three neutrinos.

   We simply convert to Joules and multiply by the number density, and by three, so we have

$$\rho_\nu = 3n_{\nu i}m_\nu = 3\times1.120\times10^8\text{ m}^{-3}\frac{\left(2\text{ eV}\right)\left(1.602\times10^{-19}\text{ J/eV}\right)}{\left(2.998\times10^8\text{ m/s}\right)^2} = 1.198\times10^{-27}\text{ kg/m}^3\,.$$

   (c) Find a maximum value for the neutrino contribution Ω_ν from neutrinos. Use the central value of the Hubble constant, H_0 = 67.8 km/s/Mpc.

   The value of Ω_ν is just $\Omega_\nu = \rho_\nu/\rho_c$, and we found ρ_c in problem set M, part 1a, so we have

$$\Omega_\nu = \frac{\rho_\nu}{\rho_c} = \frac{1.198\times10^{-27}\text{ kg/m}^3}{8.633\times10^{-27}\text{ kg/m}^3} = 0.139.$$

As we will later learn, it is rather unlikely that the neutrinos give this large a contribution, but it is interesting that it can come out to be a significant fraction.

2. **[20] Electrons and their anti-particles can be created from photon collisions,** $\gamma\gamma \to e^+e^-$**, with a cross section of approximately**

$$\sigma \approx \left(\frac{\alpha\hbar c}{E}\right)^2$$

where $\alpha = \frac{1}{137}$ **is the fine structure constant, and *E* is the typical energy of the two photons. However, this process only occurs if the energy is sufficient to make them, so that** $E > mc^2$**. The *real* formula is more complicated, but this is a good approximation.**

(a) Assume the photons have a typical energy $E = 3k_B T$**. What is the minimum temperature required to start pair creating?**

Each photon must have an energy of $E > mc^2$, and therefore $3k_B T > mc^2$, so

$$k_B T = \tfrac{1}{3}mc^2 = \tfrac{1}{3}\left(511\ \text{keV}\right) = 170\ \text{keV}$$

Since the question was worded as temperature, we should rewrite this as

$$T = \frac{k_B T}{k_B} = \frac{170{,}000\ \text{eV}}{8.617\times10^{-5}\text{eV/K}} = 1.977\times10^{9}\ \text{K}\,.$$

(b) What is the age of the universe at this time, in s? Use $g_{\text{eff}} = 3.36$**.**

As I will explain in class, it's actually ambiguous what value to use for g_{eff}, but this is a pretty good estimate. The age of the universe is

$$t = \frac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_B T}\right)^2 = \frac{2.42\ \text{s}}{\sqrt{3.36}}\left(\frac{\text{MeV}}{0.170\ \text{MeV}}\right)^2 = 46\ \text{s}$$

(c) What is the approximate cross-section at this energy?

The cross-section is:

$$\sigma \approx \left(\frac{\alpha\hbar c}{E}\right)^2 = \left(\frac{197.33\ \text{eV}\cdot\text{nm}}{137\left(511{,}000\ \text{eV}\right)}\right)^2 = 7.94\times10^{-12}\ \text{nm}^2 = 7.94\times10^{-30}\ \text{m}^2$$

This may look small, but there are a lot of photons around.

(d) What is the number density of photons at this time? Use the formula

$$n = \frac{2\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 = 0.24\left(\frac{k_B T}{\hbar c}\right)^3$$

We simply use the formula:

$$n = 0.24\left(\frac{k_B T}{\hbar c}\right)^3 = 0.24\left(\frac{170{,}000\ \text{eV}}{197.33\ \text{eV}\cdot\text{nm}}\right)^3 = 1.53\times10^{8}\ \text{nm}^{-3} = 1.53\times10^{35}\ \text{m}^{-3}$$

(e) What is the collision rate Γ for photon pairs? Assume the relative velocity is approximately $\Delta v = c$.

We should probably be using 2*c*, but never mind. The rate is

$$\Gamma = n\sigma(\Delta v) = \left(1.53\times10^{35}\ \text{m}^{-3}\right)\left(7.94\times10^{-30}\ \text{m}^{2}\right)\left(2.998\times10^{8}\ \text{m/s}\right) = 3.6\times10^{14}\ \text{s}^{-1}$$

(f) What is the number of collisions Γt? Will this process be in thermal equilibrium?

We simply multiply our previous numbers, which gives us

$$\Gamma t = \left(3.6\times10^{14}\ \text{s}^{-1}\right)\left(46\ \text{s}\right) = 1.68\times10^{16}$$

Every photon has many chances to convert to an electron positron pair, and therefore the system will easily reach equilibrium.

Graduate Problem**: Do this problem only if you are in PHY 610.**

3. [15] In problem 1, you used the current temperature of the neutrinos to find the current number density. But if neutrinos are massive, they will not be in a thermal distribution.

(a) Suppose there was a time when the neutrinos were effectively massless with a temperature T_ν. What is the typical energy for a single neutrino at such a temperature? What is the corresponding momentum p_ν?

We simply use the standard formula $\overline{E} = 3k_BT$. When neutrinos decoupled, the temperature was around an MeV, and since the masses are six orders of magnitude lighter than this, we can treat them as ultra relativistic, in which case $p = E/c$, so $\overline{p} = 3k_BT/c$.

(b) Define "T_ν" at later times so it continues to drop $\propto a^{-1}$, even when the neutrino mass can no longer be neglected (so T_ν doesn't mean temperature any more). Argue that $p_\nu \propto T_\nu$ even after this (hint – what is the wavelength for a particle with momentum *p*?)

From quantum mechanics, $p\lambda = h$. Because the universe is expanding, $\lambda \propto a$. It follows that $p \propto a^{-1}$. Since $T_\nu \propto a^{-1}$, it follows that $p \propto T_\nu$. Then, since before the electron was non-relativistic, we had $\overline{p} = 3k_BT_\nu/c$, this must remain true at all subsequent times, including now.

(c) Estimate the momentum of a typical neutrino now.

$$\overline{p} = \frac{3k_BT_\nu}{c} = \frac{3\left(8.617\times10^{-5}\text{ eV/K}\right)\left(1.945\text{ K}\right)}{c} = 5.03\times10^{-4}\text{ eV/}c.$$

(d) The heaviest neutrino has a mass between 0.05 eV/c^2 and 2 ev/c^2. Based on these two limits, find a range of typical velocities for these neutrinos. Compare with escape velocity from our galaxy (about 600 km/s).

The momentum is well below the mass, so we can use the non-relativistic formula, $p = mv$. We therefore have

$$\overline{v}_1 = \frac{\overline{p}}{m_1} = \frac{5.13\times10^{-4}\text{ eV/}c}{0.05\text{ eV/}c^2} = 1.03\times10^{-2}c = 3.08\times10^{6}\text{ m/s} = 3080\text{ km/s},$$

$$\overline{v}_2 = \frac{\overline{p}}{m_2} = \frac{5.13\times10^{-4}\text{ eV/}c}{2\text{ eV/}c^2} = 2.57\times10^{-4}c = 7.69\times10^{4}\text{ m/s} = 76.9\text{ km/s}.$$

If neutrinos are near the light end of the range they will all escape from galaxies. If they are near the upper end of the range they could be trapped in galaxies.