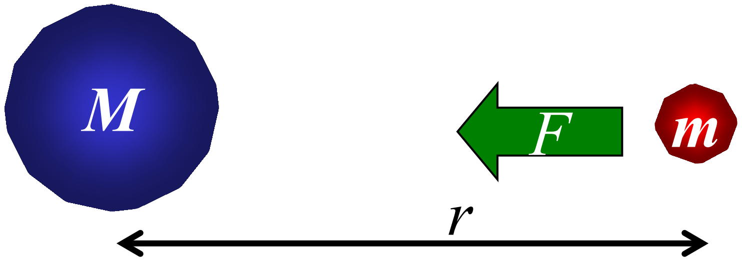
General Relativity

Principle of Equivalance

The Odd Thing about Gravity

- The force of gravity is proportional to the mass of the object it acts on

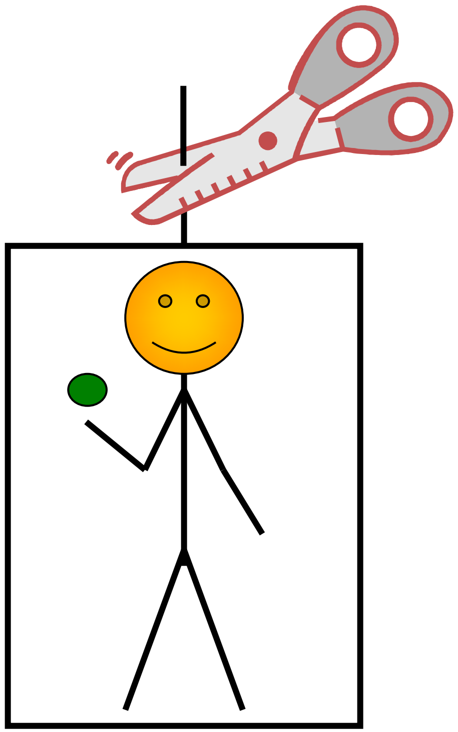
$$\vec{F} = -\frac{GMm\hat{r}}{r^2}$$



- The acceleration is the same, no matter the mass you drop

$$\vec{F} = m\vec{a} \qquad \vec{a} = -\frac{GM\hat{r}}{r^2}$$

- Suppose you are in an elevator
- Now, we cut the cable
  - You fall
  - Other things fall at the same rate
- To you, it looks like you are weightless



Weightlessness

- This is why astronauts are weightless - they are always falling
  - They are *not* far from the Earth

Principle of Equivalance

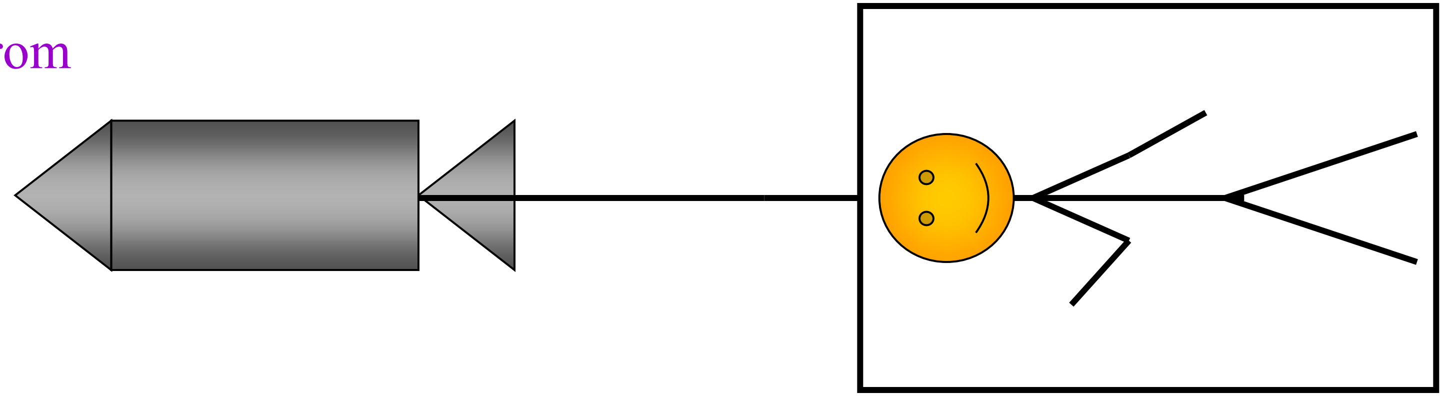
- Are there any other force formulas that are proportional to mass?
  - Suppose you are on a merry-go-round
  - Centrifugal force

$$\vec{F} = \frac{mv^2\hat{r}}{r}$$

- Like gravity, everything "accelerates" equally
- You can make it go away by letting go

Can we create "gravitational" effects through acceleration?

- Put the person out in space in the elevator
- Attach to an accelerating rocket
- The effects are indistinguishable from gravity



The effects of gravity are indistinguishable from the effects of being in an accelerated reference frame

Reexamining the Metric

- We will start by reexamining the metric – the distance formula
  - Or in this case, the proper time formula

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

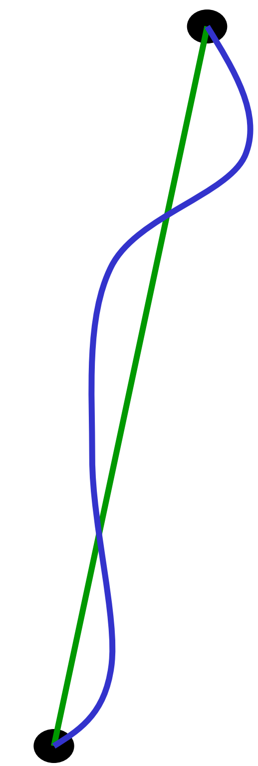
- This formula works if you move in a straight line

- What if you don't move in a straight line?
- You can divide any longer path into several short segments
- Then, find the distance formula for each one

$$c^2 d\tau^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$c\tau = \int \sqrt{c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}$$

$$c\tau = \int dt\sqrt{c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}$$



- We won't be using this formula

- Interestingly, you can show that the straight line has the *longest* proper time path – called a geodesic

An object with no forces acting on it will always follow a geodesic, which is the longest proper time path between two points in spacetime

Changing Coordinates

- It is important to be able to change coordinates to a different system

$$c^2 d\tau^2 = c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

- Let's change to spherical coordinates: (x,y,z,t) → (r,θ,ϕ,t)

$$dy = d\left(r\sin\theta\sin\phi\right) = \left(dr\right)\sin\theta\sin\phi + r\left(d\sin\theta\right)\sin\phi + r\sin\theta\left(d\sin\phi\right)$$

$$= \sin\theta\sin\phi\left(dr\right) + r\cos\theta\sin\phi\left(d\theta\right) + r\sin\theta\cos\phi\left(d\phi\right)$$

- Similarly,

$$dx = \left(dr\right)\sin\theta\cos\phi + r\left(d\sin\theta\right)\cos\phi + r\sin\theta\left(d\cos\phi\right)$$
$$= \sin\theta\cos\phi\left(dr\right) + r\cos\theta\cos\phi\left(d\theta\right) - r\sin\theta\sin\phi\left(d\phi\right)$$
$$dz = \left(dr\right)\cos\theta + r\left(d\cos\theta\right) = \cos\theta\left(dr\right) - r\sin\theta\left(d\theta\right)$$

$$c^2 d\tau^2 = \cdots = c^2 (dt)^2 - (dr)^2 - r^2 (d\theta)^2 - r^2\sin^2\theta (d\phi)^2$$

Moving in Curved Coordinates (1)

- The geodesic principle works in <u>any</u> coordinate system

An object with no forces acting on it will always follow a geodesic, which is the longest proper time path between two points in spacetime

- It is possible, starting from the metric, to find equations that describe geodesic motion

$$c^2 d\tau^2 = c^2 (dt)^2 - (dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2\theta (d\phi)^2$$

$$x^\mu = (t, r, \theta, \phi) \qquad g^{\mu\nu} = (g_{\mu\nu})^{-1}$$

$$g_{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

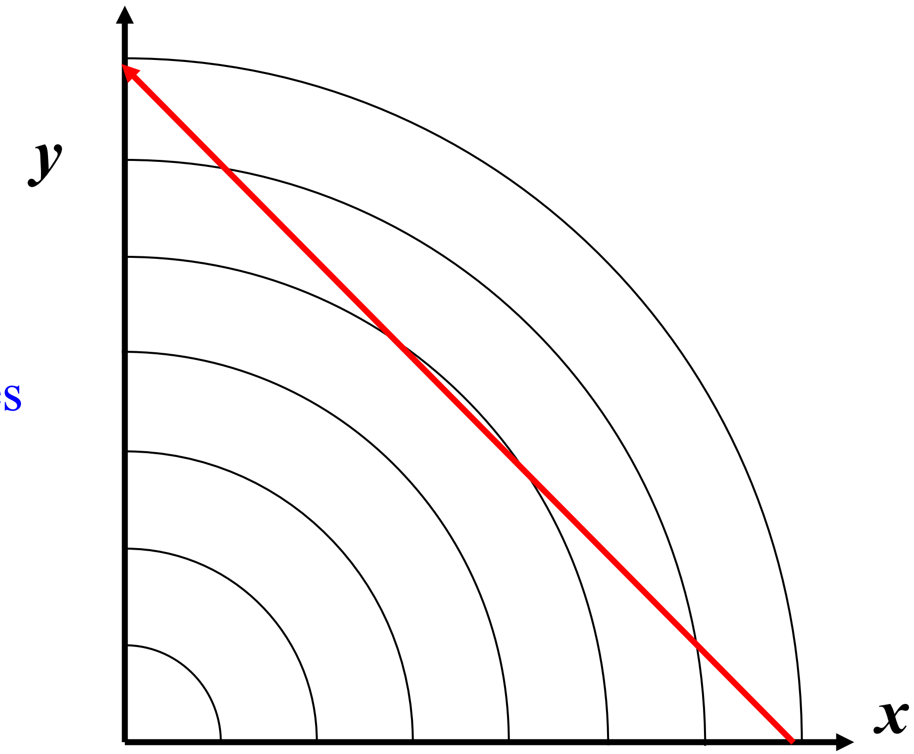
$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}\sum_\nu g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)$$

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \sum_\beta \sum_\alpha \Gamma^\mu_{\alpha\beta}\left(\frac{dx^\alpha}{d\tau}\right)\left(\frac{dx^\beta}{d\tau}\right) = 0}$$

Moving in Curved Coordinates (2)

- The effects of moving in curved coordinates looks like acceleration

$$\frac{d^2 r}{d\tau^2} = r\left(\frac{d\theta}{d\tau}\right)^2 + r\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2$$



- But it really isn't
  - It is moving as straight as it can in curved coordinates

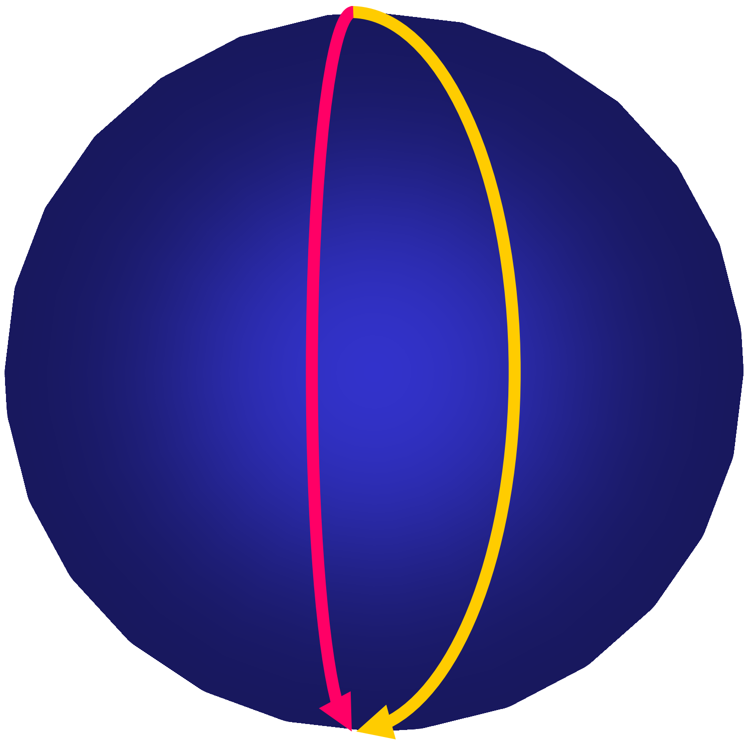
- You can always eliminate this apparent acceleration, simply by returning to "flat" coordinates

- It is possible, starting from the metric alone, to prove that spacetime is really just flat

Curved Space

- It is possible to live in a spacetime that is inherently curved
  - It's not just the coordinates, it's spacetime itself that is curved

- Consider the surface of the Earth
  - Think of it as a 2D object



Imagine two explorers setting out from the North Pole in "straight lines" (geodesics)

- At first they are traveling away from each other

- When they reach the equator, they will be traveling "parallel" to each other; their distance is no longer increasing

- They then start traveling towards each other

- They meet at the south pole

- The curvature is real

$$ds^2 = a^2 (d\theta)^2 + a^2 \sin^2 \theta (d\phi)^2$$

- It is space itself that is curved, not the coordinates only

Curvature

- How can we tell, looking only at the distance formula (the metric), if the curvature is real or a consequence of our coordinate choice?

$$R^{\mu}{}_{\nu\alpha\beta} = \frac{\partial \Gamma^{\mu}_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\mu}_{\nu\alpha}}{\partial x^{\beta}} + \sum_{\sigma} \Gamma^{\sigma}_{\nu\beta} \Gamma^{\mu}_{\alpha\sigma} - \sum_{\sigma} \Gamma^{\sigma}_{\nu\alpha} \Gamma^{\mu}_{\beta\sigma}$$

- The Riemann Tensor tells you if it is curved

- If the Riemann tensor is zero, space is not curved; if it is non-zero, it is curved

- Changing coordinates doesn't make it go away

Some other measures of curvature:

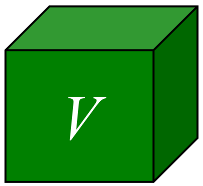
- The Ricci tensor: $$R_{\alpha\beta} = \sum_{\mu} R^{\mu}{}_{\alpha\mu\beta}$$

- The Ricci scalar: $$R = \sum_{\alpha} \sum_{\beta} R_{\alpha\beta} g^{\alpha\beta}$$

- The Einstein tensor: $$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R$$

The Stress-Energy Tensor

What causes gravity? Energy and momentum



- The presence of matter, or mass density, is the cause of gravity
  - Mass density is proportional to energy density

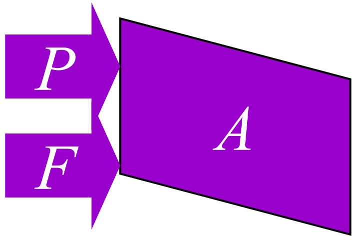
$$u = \frac{E}{V}$$

- If energy makes a difference, why not momentum as well?
  - Momentum density also contributes to gravity

$$\vec{g} = \frac{\vec{P}}{V}$$

The flow of energy and momentum also causes gravity

$$P = \vec{A} \cdot \vec{g}$$



- Another way of looking at momentum density is the transfer of energy
  - It is like power flowing through an area

$$\vec{F} = \vec{\sigma} \cdot \vec{A}$$

- You can also transmit momentum across a boundary
  - This is what forces do
  - Pressure is a good example
  - A more general example is called the stress tensor

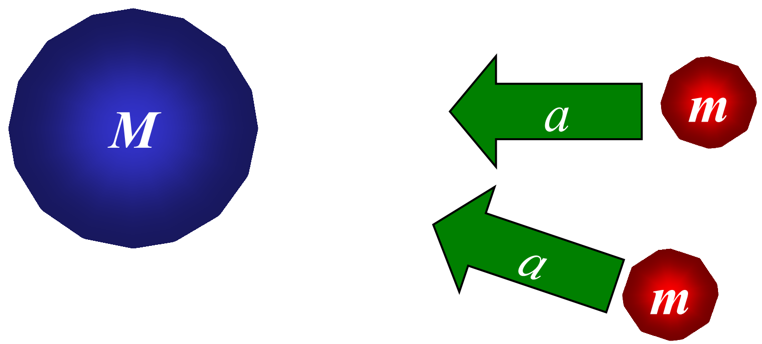
$$T_{\alpha\beta} = \begin{pmatrix} uc^2 & g_x & g_y & g_z \\ g_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ g_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ g_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

- Gravitational effects in general relativity are caused by the energy density u, the momentum density vector g, and the stress tensor σ

Einstein's Equations

The central idea of Einstein:

- Gravity looks just like acceleration, except
  - When you have a source of gravity, parallel doesn't remain parallel



- This tells you gravity has to do with curvature
- The stress-energy tensor $T_{\mu\nu}$ must be related to the curvature
- Einstein found a relationship that worked, now called Einstein's equations

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

- It may look simple, but it isn't
  - This is really 16 equations (since α and β each take on 4 values)
  - The expression on the left contains hundreds of terms
  - It is highly non-linear

Geodesics in Curved Spacetime

- Why do particles curve under the influence of gravity?
  - Because spacetime itself is curved!

An object with no non-gravitational forces acting on it will always follow a geodesic, which is the longest proper time path between two points in spacetime

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\frac{d^2 x^\mu}{d\tau^2} + \sum_\beta \sum_\alpha \Gamma^\mu_{\alpha\beta} \left(\frac{dx^a}{d\tau}\right)\left(\frac{dx^\beta}{d\tau}\right) = 0$$

Matter tells space how to curve, and space tells matter how to move

Homogenous and Isotropic Geometries

Flat Geometry in Spherical Coordinates

- The universe is, apparently, homogenous and isotropic
  - The same everywhere and in all directions
- This is sufficient to *almost* specify the shape of space right now
- What possible shape of space are uniform and isotropic?
- One obvious one is flat space, which has distance formula

$$ds^2 = dx^2 + dy^2 + dz^2$$

- Most useful to rewrite this in spherical coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

- With some work (PHY 215), the local distance formula is then

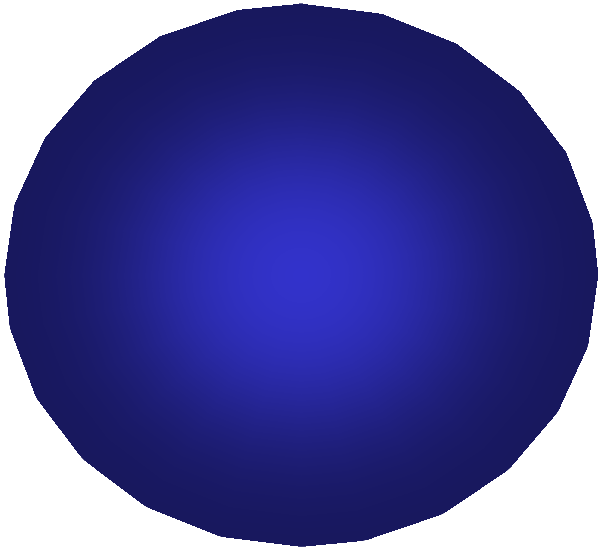
$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

The 2-Sphere

- There are other geometries that are homogenous and isotropic
- For example, the set of points a distance *a* from the origin in 3D is called a *2-sphere*

- Every point on the sphere is equivalent to every other point
- These points satisfy
- All such points can be described by giving just two angles

$$x^2 + y^2 + z^2 = a^2$$



$$x = a\sin\theta\cos\phi \qquad dx = a\cos\theta\cos\phi\, d\theta - a\sin\theta\sin\phi\, d\phi$$

$$y = a\sin\theta\sin\phi \qquad dy = a\cos\theta\sin\phi\, d\theta + a\sin\theta\cos\phi\, d\phi$$

$$z = a\cos\theta \qquad dz = -a\sin\theta\, d\theta$$

- The distance can then be worked out using the 3D flat metric

$$ds^2 = dx^2 + dy^2 + dz^2 = \cdots = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- But we want a space with three internal dimensions

The 3-Sphere

- Consider the set of points in four dimensions a distance *a* from the origin

$$x^2 + y^2 + z^2 + w^2 = a^2$$

- These points can form a three-dimensional space –
  the 3-sphere – which can be descried in terms of three angles

$$x = a\sin\psi\sin\theta\cos\phi, \quad y = a\sin\psi\sin\theta\sin\phi, \quad z = a\sin\psi\cos\theta, \quad w = a\cos\psi$$

- If you work out the distance between nearby points, it works out to

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 = \cdots = a^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

- Just like flat space, this three-dimensional space is isotropic and homogenous

Isotropic and Homogenous Geometries

- So far we have two possible geometries that work:

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

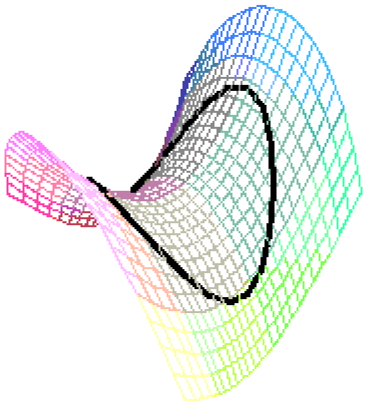
$$ds^2 = a^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

- There is one parameter describing the second one, the radius *a* or scale factor of the universe
  - In contrast, flat space has no "scale" to it
- The first is said to have zero curvature, the second has positive curvature
- There is one more type of solution, with negative curvature:

$$ds^2 = a^2\left[d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$\sinh\psi = \tfrac{1}{2}\left(e^{\psi} - e^{-\psi}\right)$$

- This one also has a scale factor *a* which is meaningful
- The 2d equivalent of this is saddle shaped



A Common Way of Writing Them

- Here are all our possible geometries:

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$ds^2 = a^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$ds^2 = a^2\left[d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

- To make them all look as similar as possible, we can make the following substitutions in cases 2 and 3:

$$r = \sin\psi \quad \text{or} \quad r = \sinh\psi$$

- Then we have

$$dr = \cos\psi d\psi = \sqrt{1-\sin^2\psi}\, d\psi = \sqrt{1-r^2}\, d\psi$$

$$dr = \cosh\psi\, d\psi = \sqrt{1+\sinh^2\psi}\, d\psi = \sqrt{1+r^2}\, d\psi$$

- And our metric becomes, in these two cases:

$$ds^2 = a^2\left[\frac{dr^2}{1\mp r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

- In fact, if you simply set $a = 1$ for the first case, you can make all three formulas nearly identical:

$$ds^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

- We have $k = 0$ for the first case, $k = +1$ for the second and $k = -1$ for the third.

It's Time to Include Time

$$ds^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \qquad k = 0,1,-1$$

- This is what the universe looks like at any moment of time

- From isotropy and homogeneity we know the only thing that can happen over time is that the distances between objects can increase or decrease uniformly
- Therefore, the only thing that is different at different times is to change *a* to a function of time

$$ds^2 = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

- We also should include time itself in the formula
- By homogeneity, time runs at the same rate everywhere
- At some point (say here), this would add a $-c^2dt^2$ to the distance formula

- Therefore, the distance formula is

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

Two Equivalent Ways of Writing the Metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$k = 0, 1, -1$$

- We can also go back to writing this equation in terms of ψ

$$ds^2 = -c^2 dt^2 + a^2(t)\left[d\psi^2 + f(\psi)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$f(\psi) = \begin{cases} \sinh\psi & \text{if } k = -1 \\ \psi & \text{if } k = 0 \\ \sin\psi & \text{if } k = -1 \end{cases}$$

- General Relativity works in any choice of coordinates
  - Pick whichever is easiest for computations

The Friedman-Robertson-Walker Metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$k = 0, 1, -1$$

- This is the only possible universe that is isotropic and homogenous at all times
- Any particular object "at rest" in this universe will stay at constant r, θ, and ϕ
- The behavior of the scale factor $a(t)$ will depend on gravity
- Einstein's equations relate the curvature $G_{\alpha\beta}$ to the presence of that which causes gravity, the Stress-Energy Tensor $T_{\alpha\beta}$:

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

- We need to find $T_{\alpha\beta}$

$$T_{\alpha\beta} = \begin{pmatrix} uc^2 & g_x & g_y & g_z \\ g_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ g_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ g_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Matter and Energy in the Universe

The Stress-Energy Tensor

- Use the fact that the universe is homogenous
  - So it can't depend on location
- And it's isotropic
  - So it can't prefer any particular direction

$$T_{\alpha\beta} = \begin{pmatrix} uc^2 & g_x & g_y & g_z \\ g_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ g_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ g_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

- The component u is just the energy density
- Since there is no net motion, this is just
  - Where ρ is the mass density

$$u = \rho c^2$$

- The components g_x, g_y, g_z represents momentum in the x-, y-, and z-direction
  - Clearly this must vanish
- Terms like σ_{xy} represents force in the y-direction transported in the x-direction
  - Terms like this must also vanish
- Terms like σ_{xx} represents pressure in the x-direction
  - These won't vanish, but must be the same in all directions
- This is just the pressure

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = P$$

Friedman's Equation

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$T_{\alpha\beta} = \begin{pmatrix} \rho c^4 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

- We now need to use Einstein's equations

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

- Getting the Einstein tensor is a lot of work
  - Fortunately, I have programs that do the work for me
- I will use only the time-time component

$$G_{\hat{t}\hat{t}} = \frac{3kc^2}{a^2} + \frac{3\dot{a}^2}{a^2}$$

- Substitute this in

$$\frac{3kc^2}{a^2} + \frac{3\dot{a}^2}{a^2} = \frac{8\pi G}{c^4}\rho c^4$$

- Now rearrange slightly
- This is the Friedman Equation we found before

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

Ω and The Shape of the Universe

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2}$$

$$\frac{\dot{a}}{a} = H$$

- Recall again that
- We also defined the density parameter

$$\Omega = \frac{\frac{8}{3}\pi G\rho}{H^2}$$

- So that we have

$$H^2 = H^2\Omega - \frac{kc^2}{a^2}$$

$$\frac{kc^2}{a^2} = H^2(\Omega - 1)$$

- If we know the sign of Ω – 1, we also know the sign of *k*
- And since *k* = 0, +1, or –1, we know *k*

Dens.	Curv.	Name
$\Omega < 1$	$k = -1$	Open
$\Omega = 1$	$k = 0$	Flat
$\Omega > 1$	$k = +1$	Closed