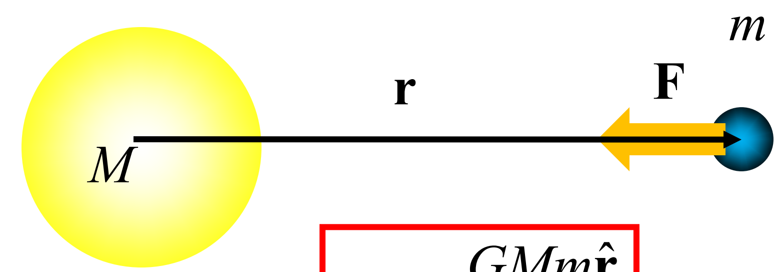
Gravity and Orbits

Gauss's Law for Gravity

Potential Energy from Gravity

The gravitational force between two objects:

- The "*r*-hat" is a unit vector pointing directly away from the source of gravity



$$\mathbf{F} = -\frac{GMm\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{F} = -\frac{GMm\left(\mathbf{r}-\mathbf{r}_M\right)}{\left|\mathbf{r}-\mathbf{r}_M\right|^3}$$

Gravitational Potential Energy

- The potential energy is the (negative of the) integral of the force

$$E_P = -\int F dr = \int \frac{GMm dr}{r^2}$$

$$E_P = -\frac{GMm}{r}$$

Simple Orbits

Circular orbits:

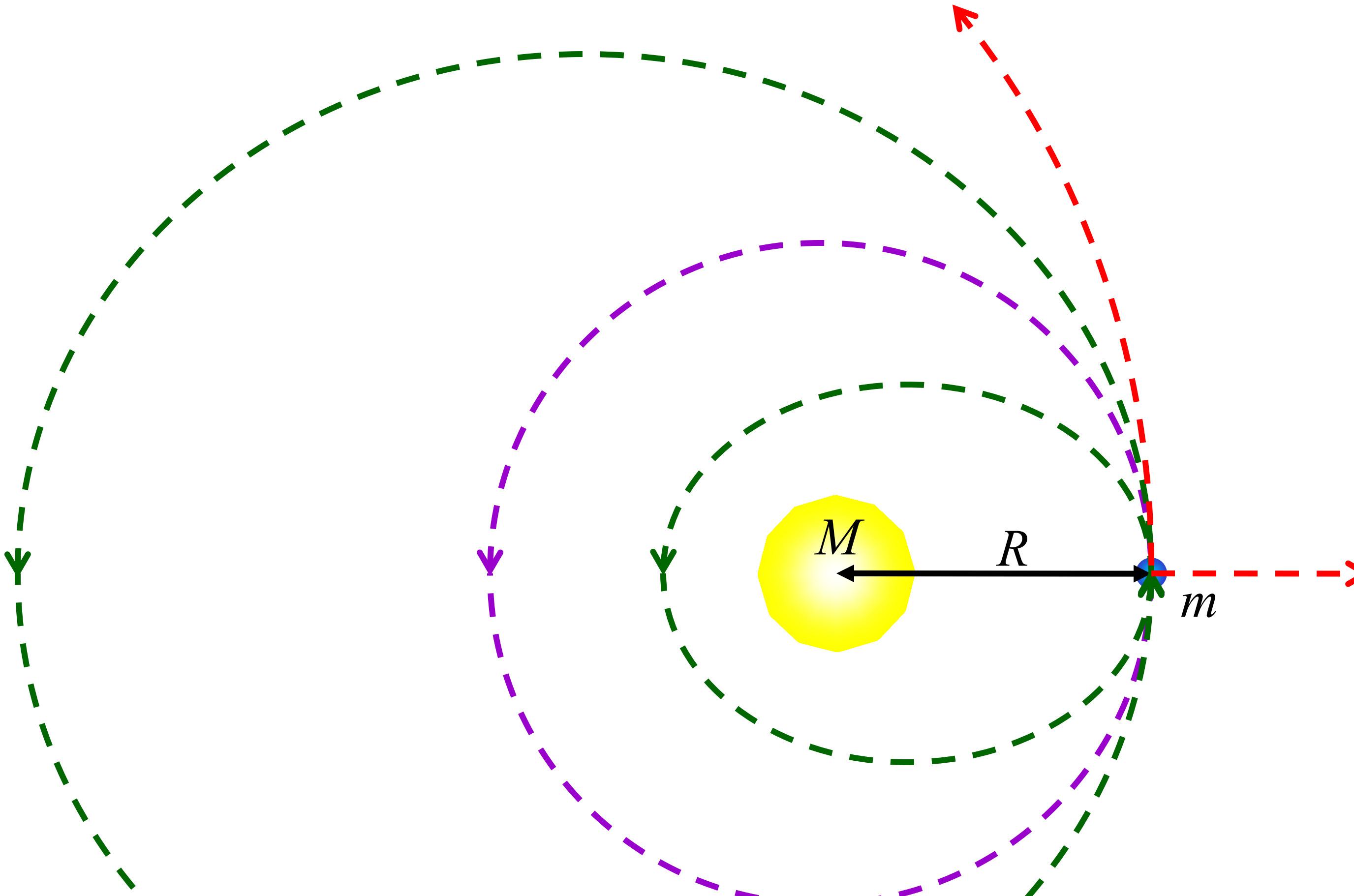
- If the velocity is exactly right, you get a circular orbit
- Gravitational force must match centripetal force

$$F = \frac{GMm}{R^2} = \frac{mv^2}{R}$$

$$v^2 = \frac{GM}{R}$$

Other orbits:

- If the velocity is smaller, or a bit bigger, you get an elliptical orbit
- If the velocity is a lot greater, the object leaves
  - Depends only on speed, not direction
- Kinetic energy must overcome potential energy
- Minimum speed is *escape velocity*



$$E_K = \tfrac{1}{2}mv^2 \geq \frac{GMm}{R}$$

$$v_{\text{esc}}^2 = \frac{2GM}{R}$$

Gravitational Field

- In PHY 114, you learned a lot about electric forces and fields
- We introduced the *Electric Field* as the force per unit charge

$$\mathbf{F} = \frac{k_e Q q \hat{\mathbf{r}}}{r^2} \qquad \mathbf{E} = \mathbf{F}/q \qquad \mathbf{E}(\mathbf{r}) = \frac{k_e Q \hat{\mathbf{r}}}{r^2}$$

- Compare this with gravity:
- By analogy, introduce the *gravitational field*

$$\mathbf{F} = -\frac{GMm\hat{\mathbf{r}}}{r^2} \qquad \boxed{\mathbf{g} = \mathbf{F}/m} \qquad \boxed{\mathbf{g}(\mathbf{r}) = -\frac{GM\hat{\mathbf{r}}}{r^2} = -\frac{GM\mathbf{r}}{|\mathbf{r}|^3}} \qquad \mathbf{F} = \mathbf{g}m = m\mathbf{a}$$

$$\boxed{\mathbf{a} = \mathbf{g}}$$

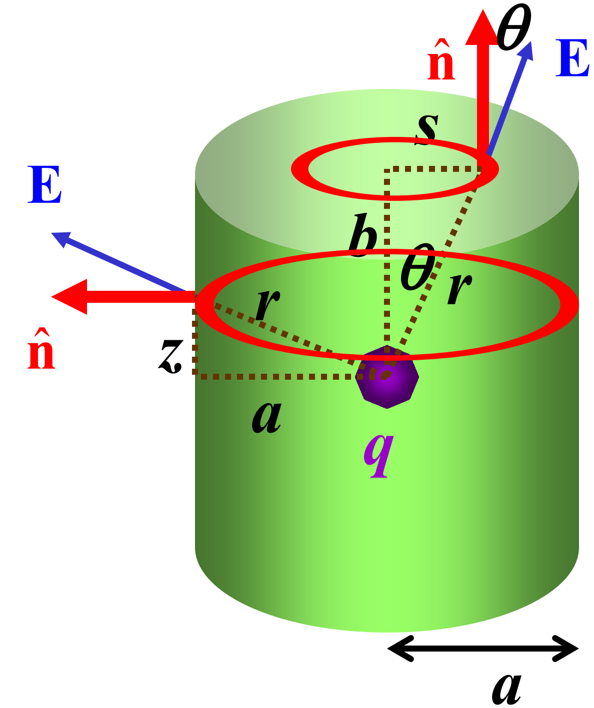
- If there are many sources of gravity, their effects must be added up

$$\boxed{\mathbf{g}(\mathbf{r}) = -\sum_i \frac{Gm_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}}$$

Gauss's Law for Gravity

- In 114, you learned the total electric flux out of a region was related to the total charge in that region:

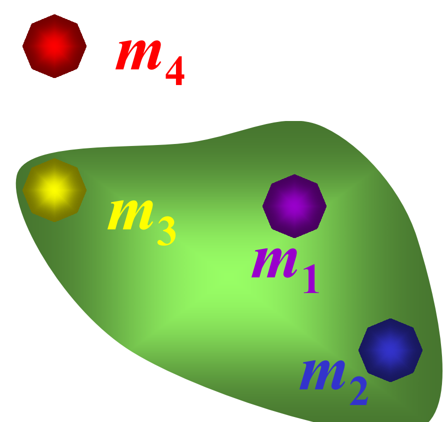
$$\Phi_E = \int \mathbf{E}\cdot\hat{\mathbf{n}}\,dA \qquad \int \mathbf{E}\cdot\hat{\mathbf{n}}\,dA = 4\pi k_e q$$



- There is an exactly analogous formula for the gravitational field:

$$\Phi_g = \int \mathbf{g}\cdot\hat{\mathbf{n}}\,dA = -4\pi GM$$

What's the gravitational flux from the region in this case?

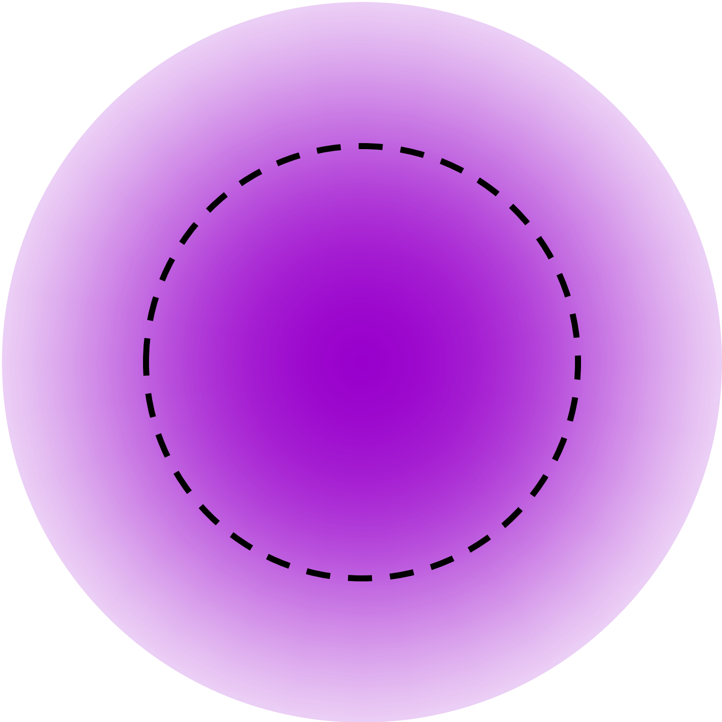


$$\Phi_g = -4\pi G\left(m_1 + m_2 + m_3\right)$$

Gauss's Law and Spherical Symmetry

- Gauss's Law can be used to find the gravitational field when there is a lot of symmetry
- Example: Spherical symmetry
  - Mass density depends only on distance from center

$$\rho(\mathbf{r}) = \rho(r)$$



- Draw a spherical Gaussian surface
- Logically, gravitational field is radial everywhere
- Gauss's Law tells you flux is proportional to contained mass

$$\Phi_g(R) = -4\pi G M(R) \qquad \Phi_g(R) = g(R)\,4\pi R^2$$

$$\mathbf{g}(R) = -\frac{GM(R)\hat{\mathbf{r}}}{R^2}$$

- The mass contained inside the sphere is just the sum of the masses on each spherical shell inside it
- The volume of a thin spherical shell is the area of a sphere of radius *r* times the thickness *dr*.

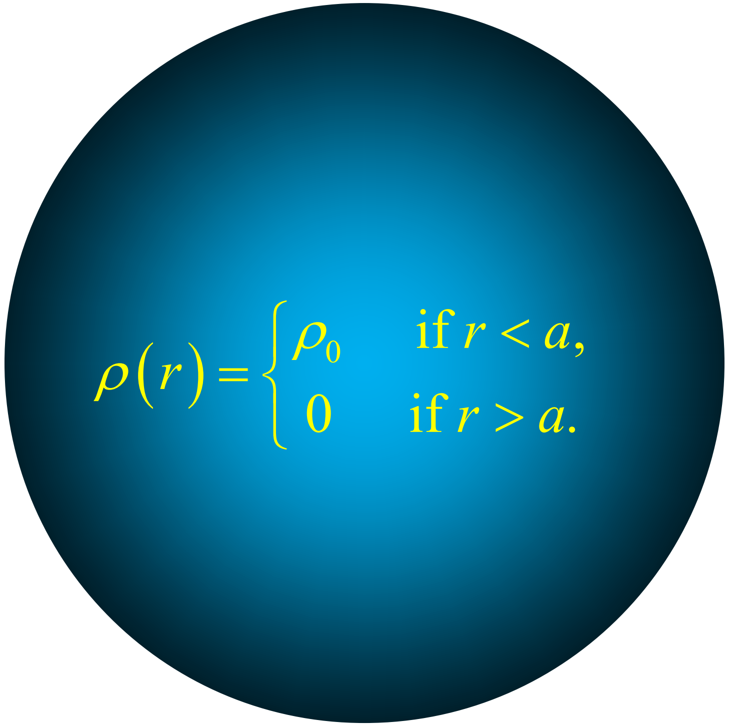
$$M(R) = \int_0^R dM = \int_0^R \rho(r)\,dV = \int_0^R 4\pi r^2 \rho(r)\,dr \qquad \mathbf{g}(R) = -\frac{G\hat{\mathbf{r}}}{R^2}\int_0^R 4\pi r^2 \rho(r)\,dr$$

Sample Problem

A gravitational source takes the form of a uniform sphere of density ρ_0 and radius a

(a) What is the gravitational field everywhere?

(b) What is the corresponding orbital velocity for circular orbits?



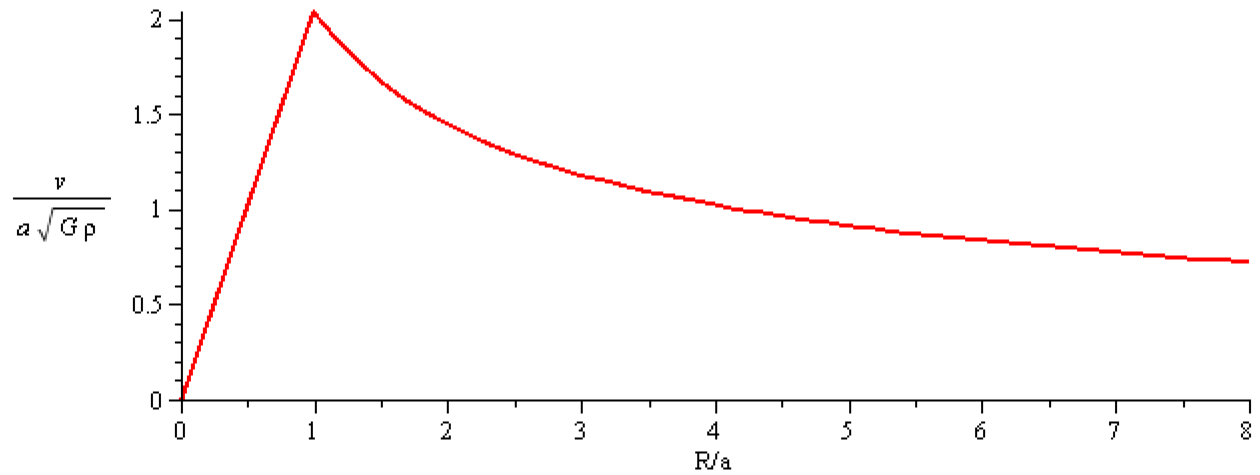
$$\mathbf{g}(R) = -\frac{G\hat{\mathbf{r}}}{R^2}\int_0^R 4\pi r^2 \rho(r)\,dr = -\frac{G\hat{\mathbf{r}}}{R^2}\int_0^{\min(a,R)} 4\pi r^2 \rho_0\,dr$$

$$= -\frac{G\hat{\mathbf{r}}\rho_0}{R^2}\frac{4\pi}{3}r^3\Big|_0^{\min(a,R)}$$

$$\mathbf{g} = \begin{cases} -\frac{4}{3}\pi G\rho_0\hat{\mathbf{r}}R & \text{if } R < a, \\ -\frac{4}{3}\pi G\rho_0\hat{\mathbf{r}}a^3R^{-2} & \text{if } R > a. \end{cases}$$

$$\mathbf{a} = \mathbf{g} = -\frac{\hat{\mathbf{r}}v^2}{R} \qquad v^2 = Rg$$

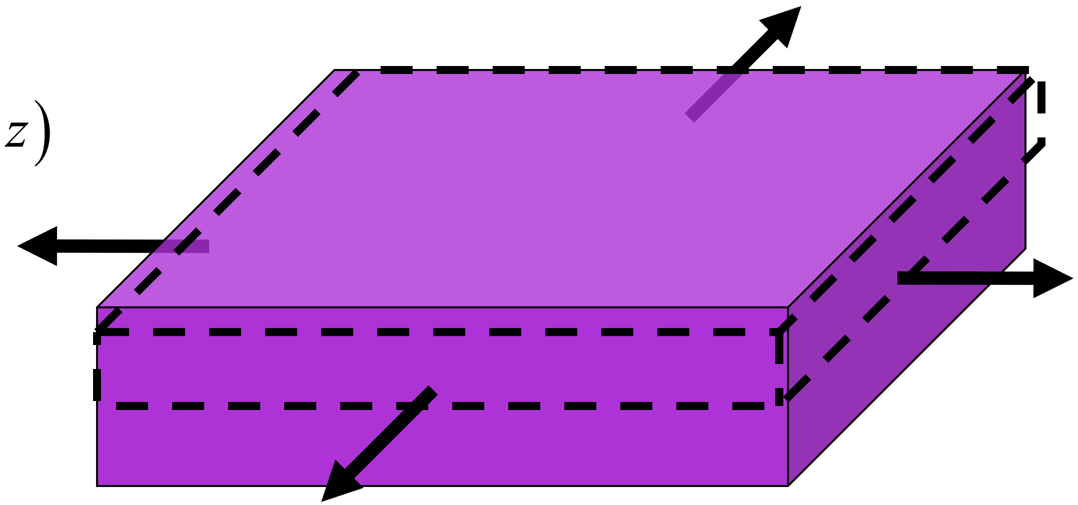
$$v = \begin{cases} \sqrt{\frac{4}{3}\pi G\rho_0}\,R & \text{if } R < a, \\ \sqrt{\frac{4}{3}\pi G\rho_0 a^3}\,R^{-1/2} & \text{if } R > a. \end{cases}$$



Gauss's Law For a Slab

- Consider a slab source, spread out uniformly in two dimensions
  - Density depends only on z.
- Assume the top half has same mass distribution as the bottom half
  - No gravity at $z = 0$.
- Draw Gaussian surface
  - Box from $z = 0$ to $z = Z$, of area A.
- Use Gauss's Law:

$$\rho(\mathbf{r}) = \rho(z)$$



$$\Phi_g(Z) = -4\pi G M(Z)$$

$$\Phi_g(Z) = g(Z)A$$

$$\mathbf{g}(Z) = -\frac{4\pi G M(Z)\hat{\mathbf{z}}}{A}$$

$$M(Z) = \int_0^Z dM = \int_0^Z \rho(z)\,dV = \int_0^Z A\rho(z)\,dz$$

$$\boxed{\mathbf{g}(Z) = -4\pi G\hat{\mathbf{z}}\int_0^z \rho(z)\,dz}$$

- For uniform density, we find:

$$\boxed{\mathbf{g}(Z) = -4\pi G\rho_0 Z\hat{\mathbf{z}}}$$

Gravitational Potential

- For electric fields, it is often more useful to work with the electrostatic potential

$$V_E(\mathbf{r}) = -\int \mathbf{E}\cdot d\mathbf{s} \qquad \mathbf{E} = -\nabla V_E(\mathbf{r}) = -\frac{\partial V_E}{\partial x}\hat{\mathbf{x}} - \frac{\partial V_E}{\partial y}\hat{\mathbf{y}} - \frac{\partial V_E}{\partial z}\hat{\mathbf{z}} \qquad V_E(\mathbf{r}) = \sum_i \frac{k_e q_i}{|\mathbf{r}-\mathbf{r}_i|}$$

- Exactly the same thing can be done for the gravitational field

$$\Phi(\mathbf{r}) = -\int \mathbf{g}\cdot d\mathbf{s} \qquad \mathbf{g} = -\nabla \Phi(\mathbf{r}) = -\frac{\partial \Phi}{\partial x}\hat{\mathbf{x}} - \frac{\partial \Phi}{\partial y}\hat{\mathbf{y}} - \frac{\partial \Phi}{\partial z}\hat{\mathbf{z}}$$

$$\Phi(\mathbf{r}) = -\sum_i \frac{G m_i}{|\mathbf{r}-\mathbf{r}_i|}$$

- The potential energy for one particle, then, is

$$E_{Pi} = m_i \Phi(\mathbf{r}_i)$$

Sample Problem (1)

What is the gravitational potential everywhere for a uniform sphere of density ρ_0 and radius a? What is escape velocity from the center of the sphere?

- Because the gravitational field is only in the *r*- direction, the potential should depend only on *r*.
- Therefore, the relationship between Φ and $\mathbf{g}$ is:

$$\mathbf{g}(r) = \begin{cases} -\frac{4}{3}\pi G\rho_0 \hat{\mathbf{r}} r & \text{if } r < a, \\ -\frac{4}{3}\pi G\rho_0 \hat{\mathbf{r}} a^3 r^{-2} & \text{if } r > a. \end{cases}$$

$$\mathbf{g}(r) = -\frac{d\Phi}{dr}\hat{\mathbf{r}} \qquad \Phi(r) = -\int g(r)\,dr$$

Must do inside and outside separately!

- Don't forget the constant of integration!

$$\Phi_{\text{in}}(\mathbf{r}) = \int \tfrac{4}{3}\pi G\rho_0 r\,dr \qquad \Phi_{\text{in}} = \tfrac{2}{3}\pi G\rho_0 r^2 + C_{\text{in}}$$

$$\Phi_{\text{out}}(\mathbf{r}) = \int \tfrac{4}{3}\pi G\rho_0 a^3 r^{-2}\,dr \qquad \Phi_{\text{out}} = -\tfrac{4}{3}\pi G\rho_0 a^3 r^{-1} + C_{\text{out}}$$

How can we find the constants of integration?

- Potential at infinity is zero
- Potential at the boundary must be continuous

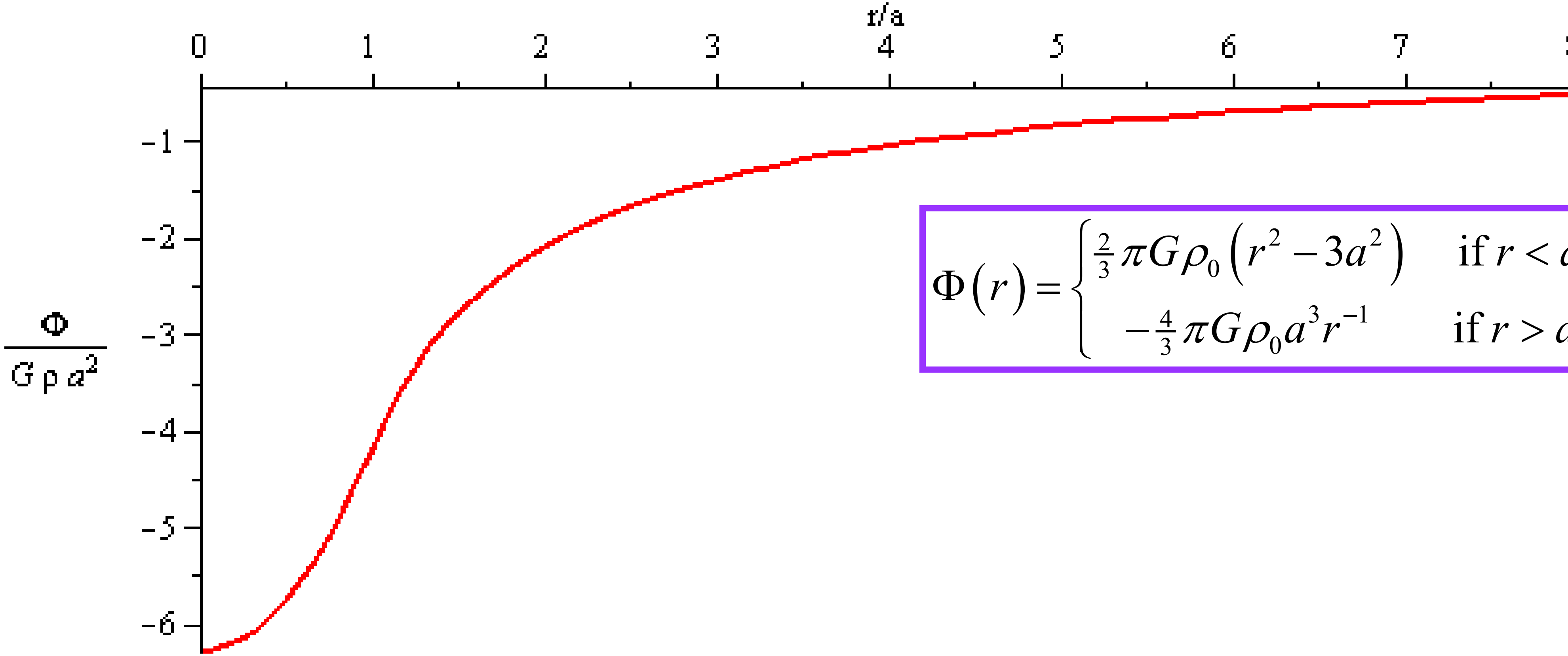
$$0 = \Phi_{\text{out}}(\infty) = 0 + C_{\text{out}} \qquad C_{\text{out}} = 0$$

$$\Phi_{\text{in}}(a) = \Phi_{\text{out}}(a)$$

$$\tfrac{2}{3}\pi G\rho_0 a^2 + C_{\text{in}} = -\tfrac{4}{3}\pi G\rho_0 a^3 a^{-1} \qquad C_{\text{in}} = -2\pi G\rho_0 a^2$$

Sample Problem (2)

What is the gravitational potential everywhere for a uniform sphere of density ρ_0 and radius a? What is escape velocity from the center of the sphere?



$$\Phi(r) = \begin{cases} \frac{2}{3}\pi G\rho_0\left(r^2 - 3a^2\right) & \text{if } r < a, \\ -\frac{4}{3}\pi G\rho_0 a^3 r^{-1} & \text{if } r > a. \end{cases}$$

- To find escape velocity, match potential and kinetic energy at the origin

$$\left|E_P\right| = m\left|\Phi(0)\right| = 2\pi G\rho_0 a^2 m = E_K = \tfrac{1}{2}mv^2$$

$$v_{\text{esc}}^2 = 4\pi G\rho_0 a^2 = 3GM/a$$

Global Conservation Laws

- Consider a galaxy/structure which is not interacting with other galaxies/structures:
  - Often a good approximation
- The following must be conserved:
  - Total energy, Total momentum, Total angular momentum

Total momentum describes overall motion

- We can eliminate it by working in the "center of mass frame"

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i$$

Energy is more complicated

  - Potential and kinetic
- But there can be other contributions
- If gravity is the only force involved, then global $E_p + E_K$ energy is conserved
  - Stars, for example, rarely collide
- If there are other effects, like collisions, energy can be transferred and lost
  - Gas clouds, in contrast, commonly collide

$$E = E_P + E_K + E_{\text{other}}$$

$$E_P = \sum_{i<j} \frac{G m_i m_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|} \qquad E_K = \sum_i \tfrac{1}{2} m_i \mathbf{v}_i^2$$

- Finally, the *total* angular momentum of the galaxy is conserved

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i$$

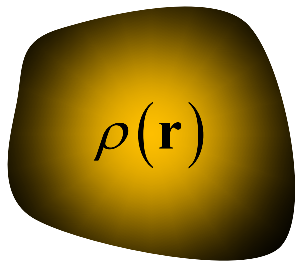
Gravitational Potential from a Distribution

- Suppose mass is distributed in a continuous manner
  - How do we calculate potential and potential energy?

$$\Phi(\mathbf{r}) = -\sum_i \frac{Gm_i}{|\mathbf{r}-\mathbf{r}_i|}$$

- Divide into many small regions of size dV
  - These will each have mass $\rho\, dV$

$$\Phi(\mathbf{r}) = -\sum_i \frac{G\rho_i(\mathbf{r}_i)\,dV_i}{|\mathbf{r}-\mathbf{r}_i|}$$



- Convert to an integral:

$$\Phi(\mathbf{r}) = -\int \frac{G\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}'$$

- Potential energy for the whole system is:

$$E_P = -\sum_{i<j} \frac{Gm_i m_j}{|\mathbf{r}_i-\mathbf{r}_j|} = -\frac{1}{2}\sum_{i\neq j} \frac{Gm_i m_j}{|\mathbf{r}_i-\mathbf{r}_j|} = -\frac{1}{2}\sum_{i\neq j} \frac{G\rho(\mathbf{r}_i)\,dV_i\,\rho(\mathbf{r}_j)\,dV_j}{|\mathbf{r}_i-\mathbf{r}_j|}$$

- Can be rewritten in terms of potential:

$$E_P = -\frac{1}{2}\int d^3\mathbf{r}\int d^3\mathbf{r}'\,\frac{G\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$E_P = \frac{1}{2}\int d^3\mathbf{r}\,\rho(\mathbf{r})\,\Phi(\mathbf{r})$$

Sample Problem

What is the total gravitational binding energy for a uniform sphere of mass *M* and radius *a*?

$$E_P = \tfrac{1}{2}\int d^3\mathbf{r}\,\rho(\mathbf{r})\Phi(\mathbf{r})$$

$$\Phi(r) = \begin{cases} \tfrac{2}{3}\pi G\rho_0\left(r^2 - 3a^2\right) & \text{if } r < a, \\ -\tfrac{4}{3}\pi G\rho_0 a^3 r^{-1} & \text{if } r > a. \end{cases}$$

$$E_P = \frac{1}{2}\cdot\frac{2}{3}\pi G\int d^3\mathbf{r}\,\rho_0\rho_0\left(r^2 - 3a^2\right) = \frac{1}{3}\pi G 4\pi\rho_0^2\int_0^a r^2 dr\left(r^2 - 3a^2\right) = \frac{4}{3}\pi^2 G\rho_0^2\left[\tfrac{1}{5}r^5 - a^2 r^3\right]_0^a$$

$$= -\frac{16}{15}\pi^2 G\rho_0^2 a^5$$

$$M = \frac{4}{3}\pi a^3\rho_0 \qquad \rho_0 = \frac{3M}{4\pi a^3}$$

$$E_P = -\frac{16\pi^2 G a^5}{15}\left(\frac{3M}{4\pi a^3}\right)^2$$

$$E_P = -\frac{3GM^2}{5a}$$

Virial Theorem (1)

- For circular orbits, there is a simple relationship between the potential energy and the kinetic energy:
- For non-circular orbits, this is not true, because energy keeps changing between the two components.
- However, if you average over time, this will still be true

$$E_P = -\frac{GMm}{r}$$

$$E_K = \tfrac{1}{2} m \mathbf{v}^2 = \tfrac{1}{2} m \frac{GM}{r}$$

$$2E_K + E_P = 0$$

$$2\langle E_K \rangle + \langle E_P \rangle = 0$$

- If you have *many* objects, some of them will be at their maximum, and others at their minimum
  - Could this expression be true if you add everything up?
- Consider a complicated combination of *many* masses acting gravitationally
  - Galaxy or Globular cluster, for example, consists of 10^4 to 10^{14} stars
- First, find the total kinetic and potential energy
- And the force on any one object

$$E_K = \sum_i \tfrac{1}{2} m \mathbf{v}_i^2$$

$$E_P = -\sum_{i<j} \frac{G m_i m_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

$$\mathbf{F}_i = -\sum_{j \neq i} \frac{G m_i m_j \left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$

Virial Theorem (2)

- We will assume that the system isn't changing much; *i.e.*, though the individual stars are moving, there will be as many moving one way as another
  - Galaxy has no net motion
  - Any quantity that "adds up" effects of all components will be constant

$$E_K = \sum_i \tfrac{1}{2} m\mathbf{v}_i^2$$

$$E_P = -\sum_{i<j} \frac{Gm_i m_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

$$\mathbf{F}_i = -\sum_{j\neq i} \frac{Gm_i m_j\left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$

- Consider the following quantity: $\sum_i m_i \mathbf{r}_i \cdot \mathbf{v}_i = \text{constant}$
- Time derivative should vanish:

$$0 = \sum_i \frac{d}{dt}\left(m_i \mathbf{r}_i \cdot \mathbf{v}_i\right) = \sum_i \left(m_i \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{v}_i + m_i \mathbf{r}_i \cdot \frac{d\mathbf{v}_i}{dt}\right) = \sum_i \left(m_i \mathbf{v}_i^2 + \mathbf{r}_i \cdot \mathbf{F}_i\right)$$

$$= 2E_K - \sum_i \sum_{j\neq i} Gm_i m_j \frac{\mathbf{r}_i \cdot \left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3} = 2E_K - \sum_{i<j} Gm_i m_j \left[\frac{\mathbf{r}_i \cdot \left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3} + \frac{\mathbf{r}_j \cdot \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^3}\right]$$

$$= 2E_K - \sum_{i<j} Gm_i m_j \frac{\left(\mathbf{r}_i - \mathbf{r}_j\right)^2}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3} = 2E_K + E_P$$

$$2E_K + E_P = 0$$

Measuring Rotation

The Sun's Revolution

Measuring rotation in our galaxy is *hard* because we are inside it.

One method for measuring circular rate of rotation at our radius:

- Study proper motion of Sagittarius A* over period of years
  - Possible using radio telescopes and interferometry
- Multiply by distance, 8.23 ± 0.12 kpc
  - Result is about 230 ± 10 km/s
- Subtract the Sun's motion compared to nearby objects (local standard of rest):
  - Sun moves forward at 12 ± 2 km/s, upwards at 7 ± 1 km/s, inwards at 11 ± 1 km/s

$$V_0 = 220 \pm 20 \text{ km/s}$$

Measuring Other Object's Revolutions (1)

For other objects, it is *hard* to measure their rotation rates

- Assume they are going in circular orbits at speed V
- Let l (galactic longitude) be angle as viewed by us
- Let α be angle of star viewed from center
- Law of cosines:
- Take time derivative: $D^2 = R^2 + R_0^2 - 2RR_0 \cos\alpha$

$$v_r = \frac{RR_0}{D}\sin\alpha\frac{d\alpha}{dt} \qquad 2D\frac{dD}{dt} = 2RR_0\sin\alpha\frac{d\alpha}{dt}$$

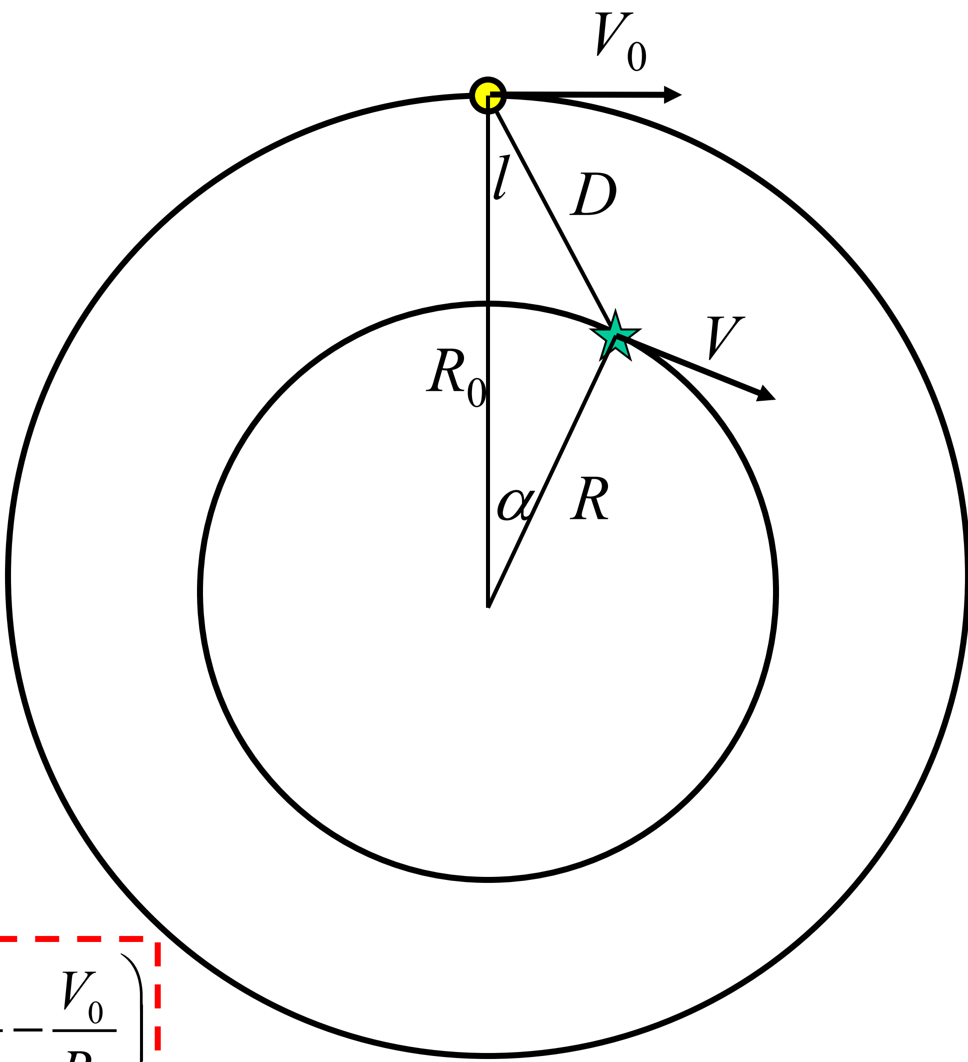
- The angular velocity of us and the other star is:

$$\frac{d\theta_0}{dt} = \frac{V_0}{R_0} \qquad \frac{d\theta}{dt} = \frac{V}{R} \qquad \frac{d\alpha}{dt} = \frac{V}{R} - \frac{V_0}{R_0}$$

$$v_r = \frac{RR_0}{D}\left(\frac{V}{R} - \frac{V_0}{R_0}\right)\sin\alpha$$

- Law of sines: $\dfrac{\sin\alpha}{D} = \dfrac{\sin l}{R}$

$$v_r = R_0 \sin l\left(\frac{V}{R} - \frac{V_0}{R_0}\right)$$



Measuring Other Object's Revolutions (2)

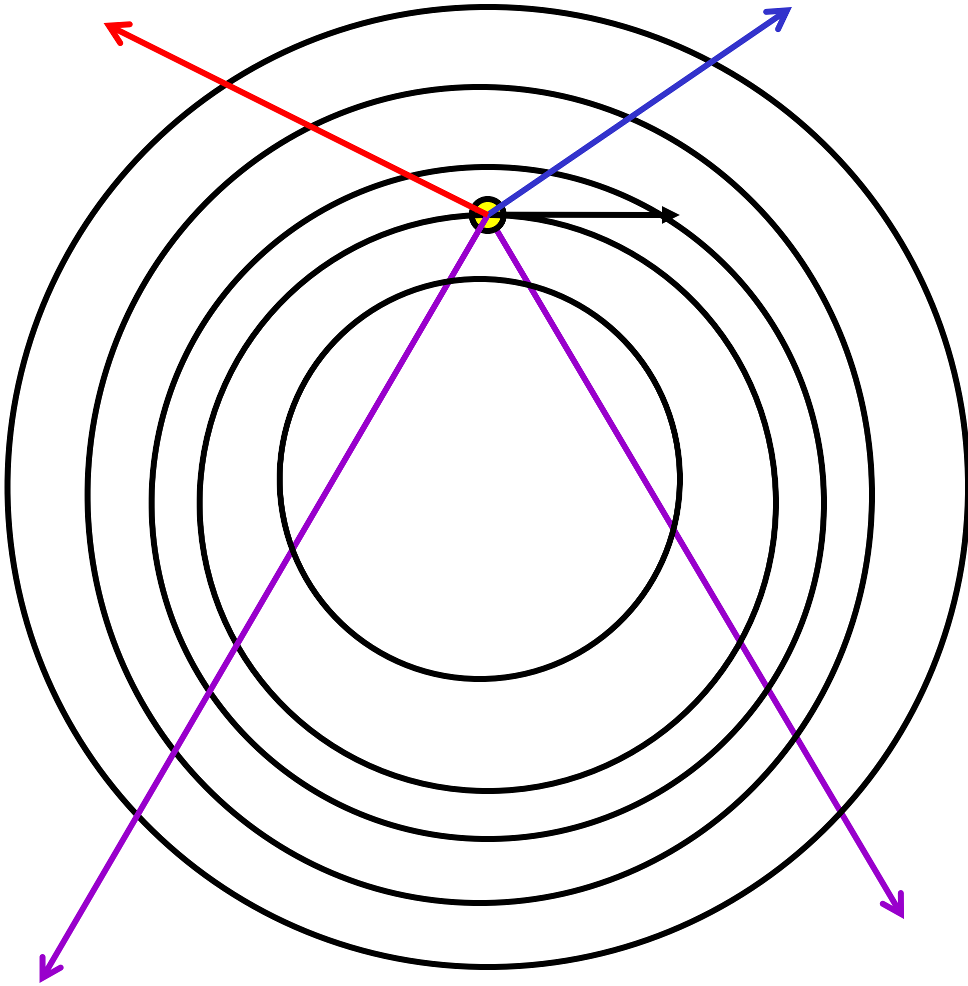
- As we look inward, we see both closer and farther orbits
  - We see a mix of red and blue shifts

- As we look outward, we see only more distant orbits
  - We see only red or blue shifts, depending on l
  - Blue shift forwards
  - Red shift backwards

- Conclusion: Gas clouds at larger radius have smaller angular velocity (V/R is smaller)

- V/R decreases with radius

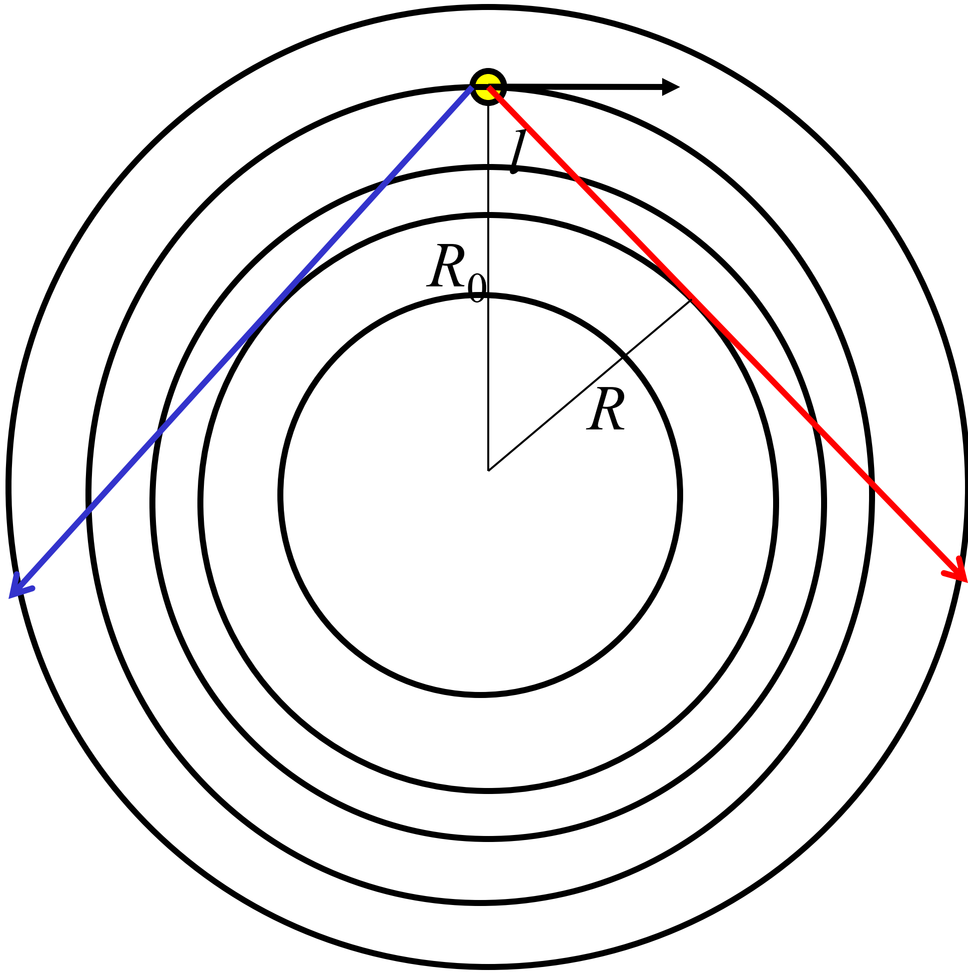
$$v_r = R_0 \sin l \left(\frac{V}{R} - \frac{V_0}{R_0} \right)$$



The Tangent Method

- Look inwards/forward at an angle
  - Clouds closer to the center will be red-shifted
  - Because they are moving at higher angular velocity
- The one closest to the center will be the most red-shifted
- The biggest Doppler red shift lets you calculate V

$$v_r = R_0 \sin l \left(\frac{V}{R} - \frac{V_0}{R_0} \right)$$



$$V = \frac{R}{R_0}\left(\frac{v_r}{\sin l} + V_0 \right) \qquad R = R_0 \sin l$$

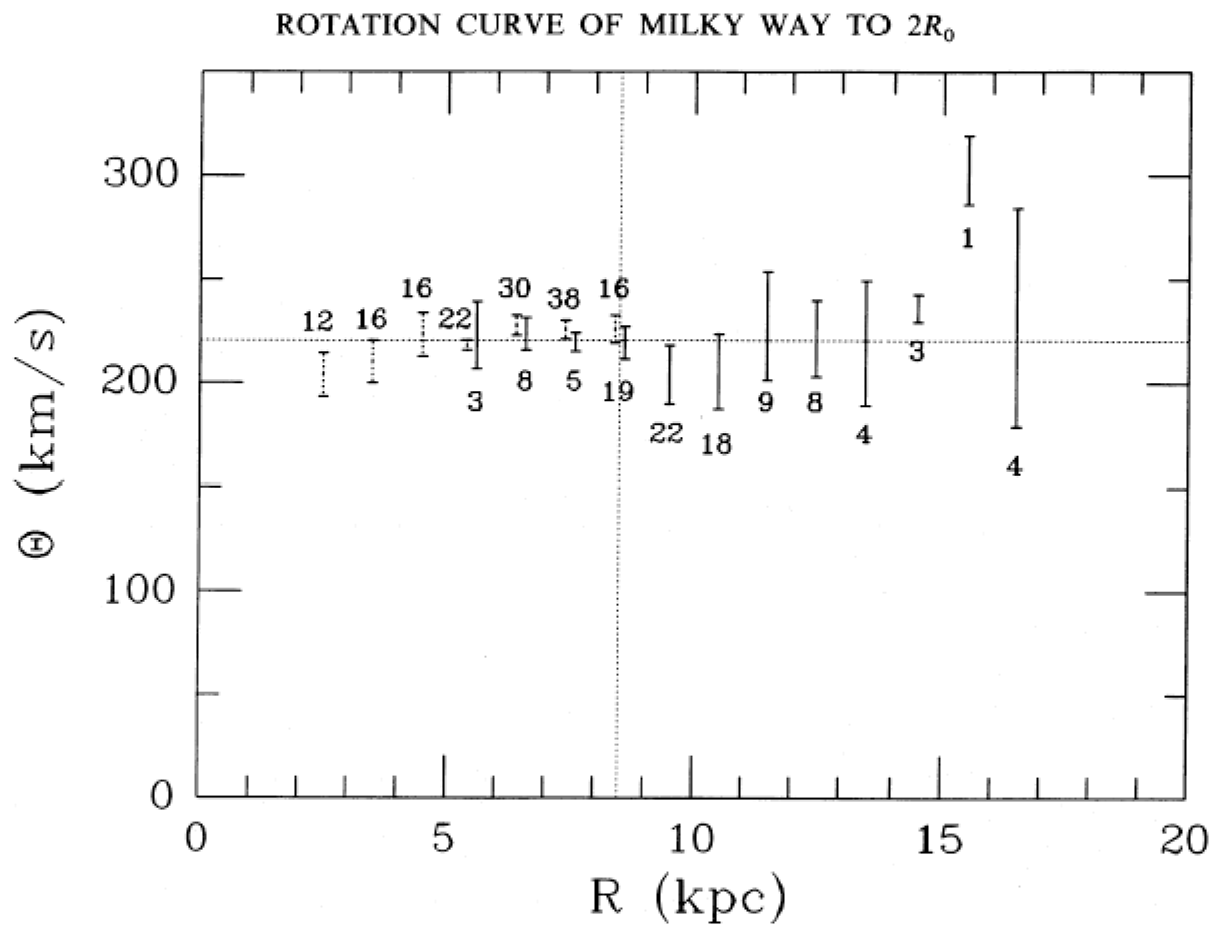
$$V\Big|_{R_0 \sin l} = v_r + V_0 \sin l$$

- In a similar manner, you can look backwards
  - Clouds closer to the center will be blue-shifted
  - Because they are moving at higher angular velocity
- The one closest to the center will be the most blue-shifted
- The biggest Doppler blue shift lets you calculate V

$$V\Big|_{-R_0 \sin l} = -v_r - V_0 \sin l$$

Rotations for Other Radii

- At small radii, the gas cloud orbits are not very circular
  - Tangent method gives inaccurate results
- Tangent method only tells you results for inner orbits
- For more distant orbits:
- Measure the distance to a star or cluster of stars
- Measure radial velocity
- Deduce orbital velocity



Rotation Curves and Mass Distribution

- Let's crudely assume the mass is distributed in a spherically symmetric manner
  - This is *not* true, but probably only introduces 10 – 20% error

$$V \approx 220 \text{ km/s}$$

$$M\left(R_0\right) \approx 93 \times 10^9 M_\odot$$

$$V^2 = \frac{GM\left(R\right)}{R}$$

- Then we can find the mass closer than the Sun using $R_0 = 8.3$ kpc
  - This is the same as the mass of *all* the stars and gas in the galaxy
  - *Suggests* some missing mass
- Stars can be seen out to at least twice this radius
  - Total mass is at least twice this
  - Mass is *not* concentrated near the center
- 21 cm line from atomic hydrogen out to at least 5 times this, maybe more

- We can also study orbits of globular clusters out past 50 kpc
- Finally, there are small galaxies orbiting ours out to 200 kpc or so
  - Again, speeds remain comparable

- 90% of the mass of the galaxy is <u>not</u> in the disk (nor the bulge), but in the halo
- Approximately spherically symmetric
- Mass contained in radius R is roughly proportional to R

Dark Matter

$V \approx 220$ km/s

$$V^2 = \frac{GM(R)}{R}$$

- The Halo contains most of the mass of our galaxy
  - Probably around 90%
- This matter is dark – it contributes little or nothing to the luminosity
- We don't know how far it goes out
  - At least 100 kpc
  - Probably less than half the distance to the next large galaxy
    - Andromeda galaxy ½(800 kpc) = 400 kpc
  - Probably around 300 kpc
- Total galaxy mass is probably around $10^{12}\ M_\odot$

Object	Mass (M_{Sun})
Disk Stars	60×10^9
Disk Gas	$\sim 10 \times 10^9$
Bulge	20×10^9
Halo Stars	1×10^9
Nucleus	0.01×10^9
Dark Matter	$> 500 \times 10^9$

What is the Dark Matter? (1)

What could the dark matter be?

- Could it be gas?
  - HI regions – produce spectral lines or X-rays – NO
  - HII regions – produce the 21 cm line – NO
  - Molecular clouds
    - Contaminants like CO produce spectral lines – NO
    - But perhaps there are clouds that are pure hydrogen – MAYBE
- Arguments based on cosmology suggest we see most of the gas that is present – PROBABLY NOT

- Could it be massive objects like:
  - White dwarfs – difficult to see since they are dim
  - Neutron stars – even harder to see
  - Black holes – impossible to see
  - “Jupiters” or “brown dwarfs” –formed without stars
- These objects are collectively called Massive Compact Halo Objects (MACHOs)

- Invisible massive particles

MACHOs and Bending of Light (1)

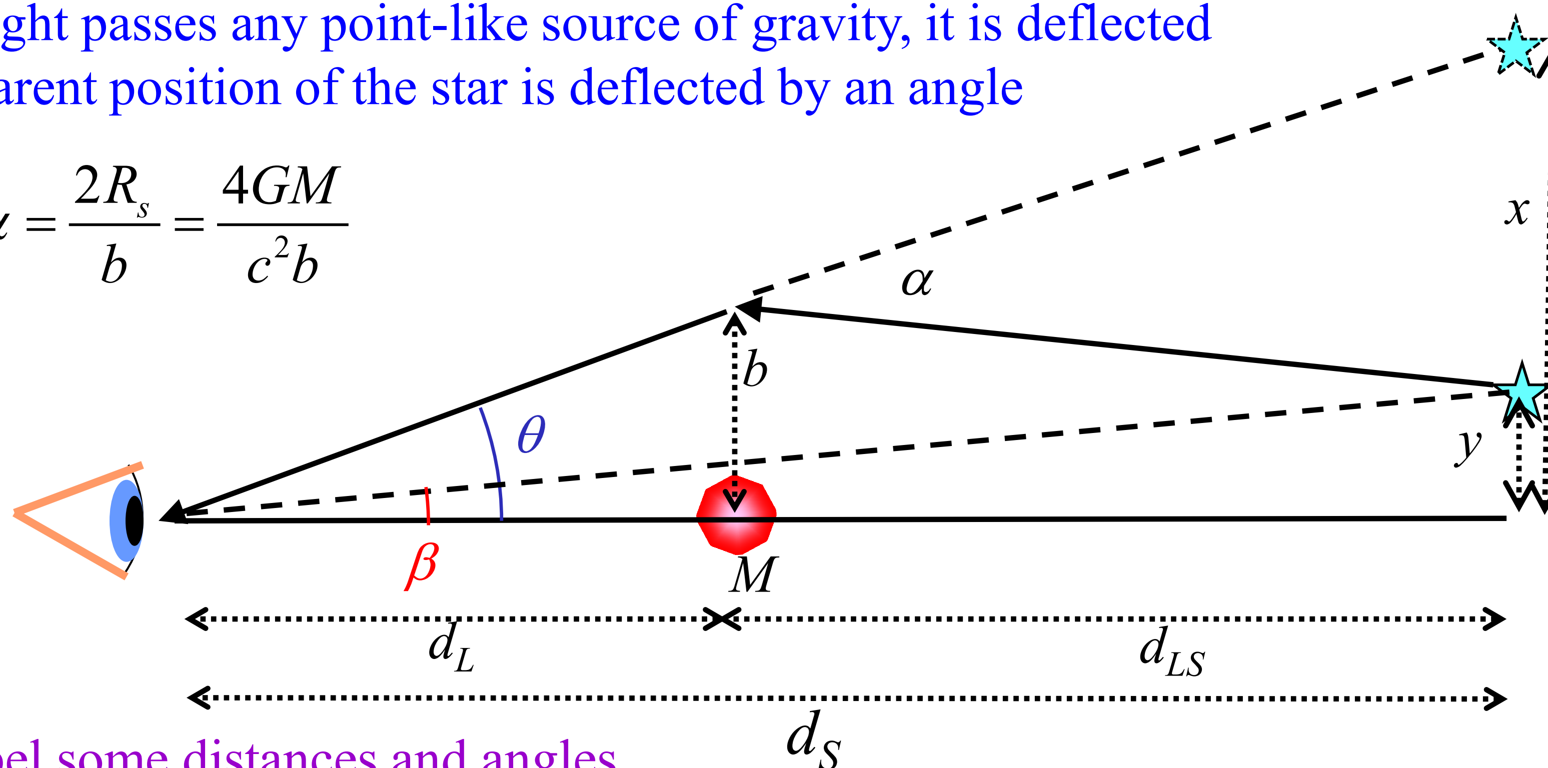
- All of these objects are dim and hard to see
- However, they all have a lot of gravity
- According to Einstein, gravity bends light

- As light passes any point-like source of gravity, it is deflected
- Apparent position of the star is deflected by an angle

MACHOS

- White dwarfs
- Neutron stars
- Black Holes
- Brown dwarfs/ "Jupiters"

$$\alpha = \frac{2R_s}{b} = \frac{4GM}{c^2 b}$$



$$\theta = \frac{b}{d_L} = \frac{x}{d_S}$$

$$\beta = \frac{y}{d_S}$$

$$\alpha = \frac{x-y}{d_{LS}}$$

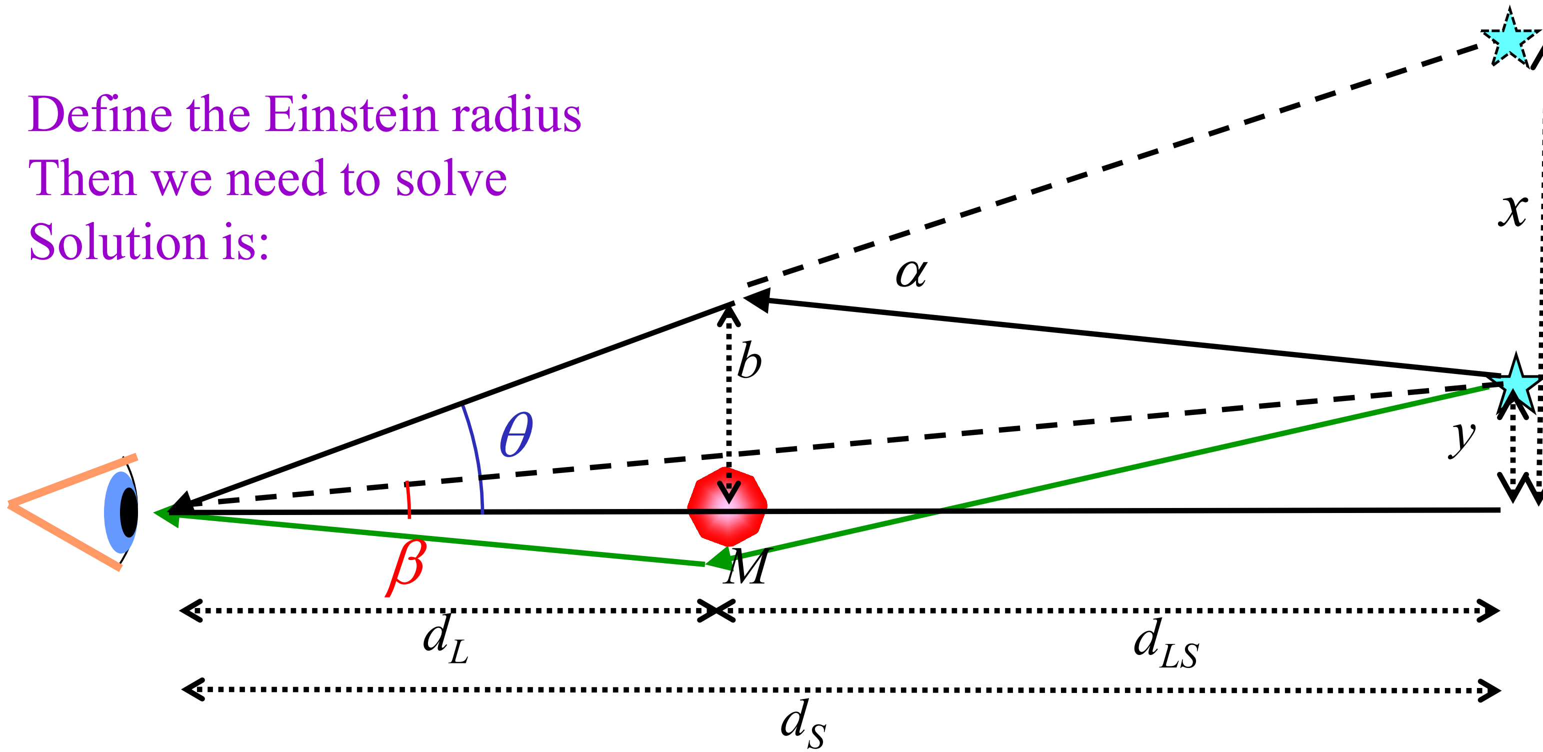
- Label some distances and angles
- Use small angle approximations

$$\theta - \beta = \frac{x-y}{d_s} = \frac{d_{LS}\alpha}{d_S} = \frac{4GMd_{LS}}{c^2 b d_S} = \frac{4GMd_{LS}}{c^2 d_L d_S \theta}$$

- Do some math

MACHOs and Bending of Light (2)

- Define the Einstein radius
- Then we need to solve
- Solution is:



$$\theta - \beta = \frac{4GMd_{LS}}{c^2 d_L d_S \theta}$$

$$\theta_E^2 \equiv \frac{4GMd_{LS}}{c^2 d_L d_S}$$

$$\theta^2 - \beta\theta - \theta_E^2 = 0$$

$$\theta = \tfrac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

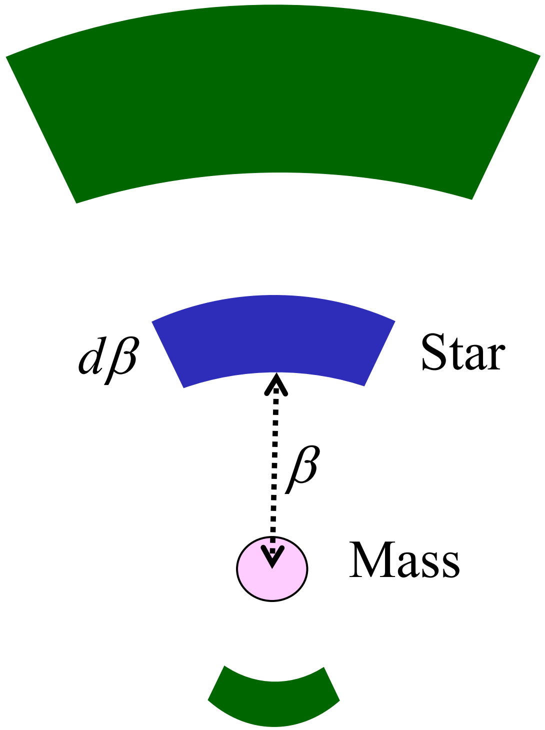
- There are, in fact, *two* images
  - One deflected above, as sketched
  - One deflected below

- Unfortunately, these angles are too small to detect
- Nonetheless, they can still magnify the star, making it brighter

MACHOs and Magnification of Light

- To make things simple, assume shape of star is part of an annulus centered on the mass
  - Any shape can be made of such annuli

$$\theta = \tfrac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$



- The star *actually* goes from angle β to $\beta + d\beta$
  - And from azimuthal angle ϕ to $\phi + d\phi$
- Without the mass, the star's brightness would be proportional to

$$F_S \propto (\beta d\phi)(d\beta)$$

- The star's *image* goes from angle θ to $\theta + d\theta$
  - And from azimuthal angle ϕ to $\phi + d\phi$
- The brightness of the image is, therefore,

$$F_I \propto (\theta d\phi)(d\theta)$$

- The ratio of these is the magnification
- The actual brightness is the sum of the two images put together

$$A_\pm = \left|\frac{\theta}{\beta}\frac{d\theta}{d\beta}\right| = \frac{1}{4}\left(\frac{\beta}{\sqrt{\beta^2 + 4\theta_E^2}} + \frac{\sqrt{\beta^2 + 4\theta_E^2}}{\beta} \pm 2\right)$$

$$A_{tot} = \frac{1}{2}\left(\frac{\beta}{\sqrt{\beta^2 + 4\theta_E^2}} + \frac{\sqrt{\beta^2 + 4\theta_E^2}}{\beta}\right)$$

How to Catch a MACHO

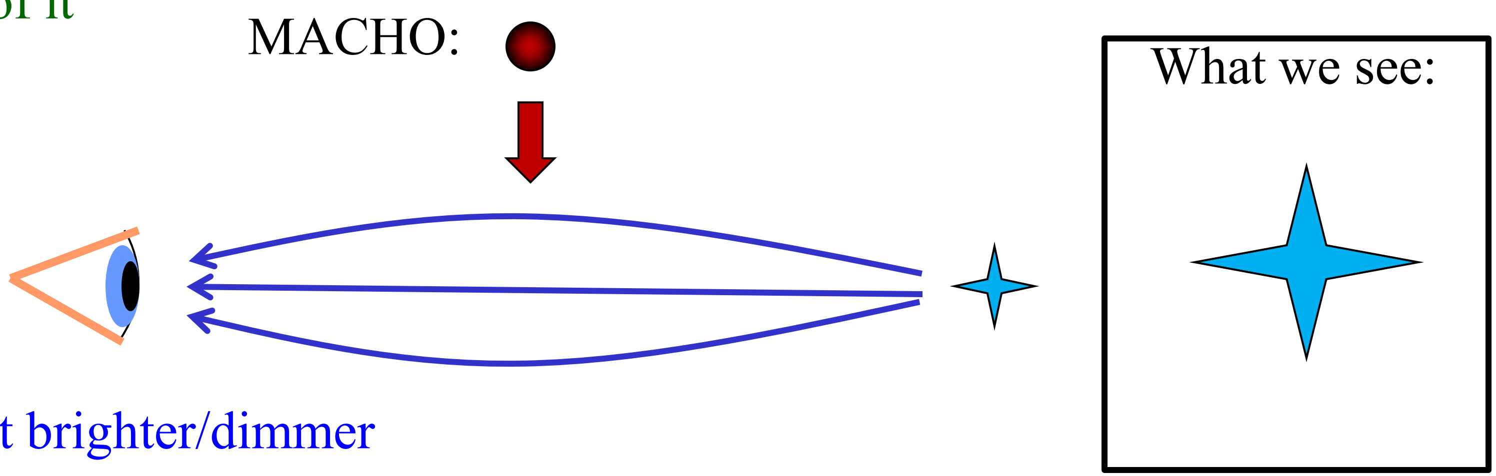
$$A_{tot} = \frac{1}{2}\left(\frac{\beta}{\sqrt{\beta^2 + 4\theta_E^2}} + \frac{\sqrt{\beta^2 + 4\theta_E^2}}{\beta} \right)$$

MACHOS

- White dwarfs
- Neutron stars
- Black Holes
- Brown dwarfs/"Jupiters"

- Watch a random star

- MACHO will pass in front of it
- Light gets bent
- Star gets brighter
- MACHO moves away
- Star gets dimmer again

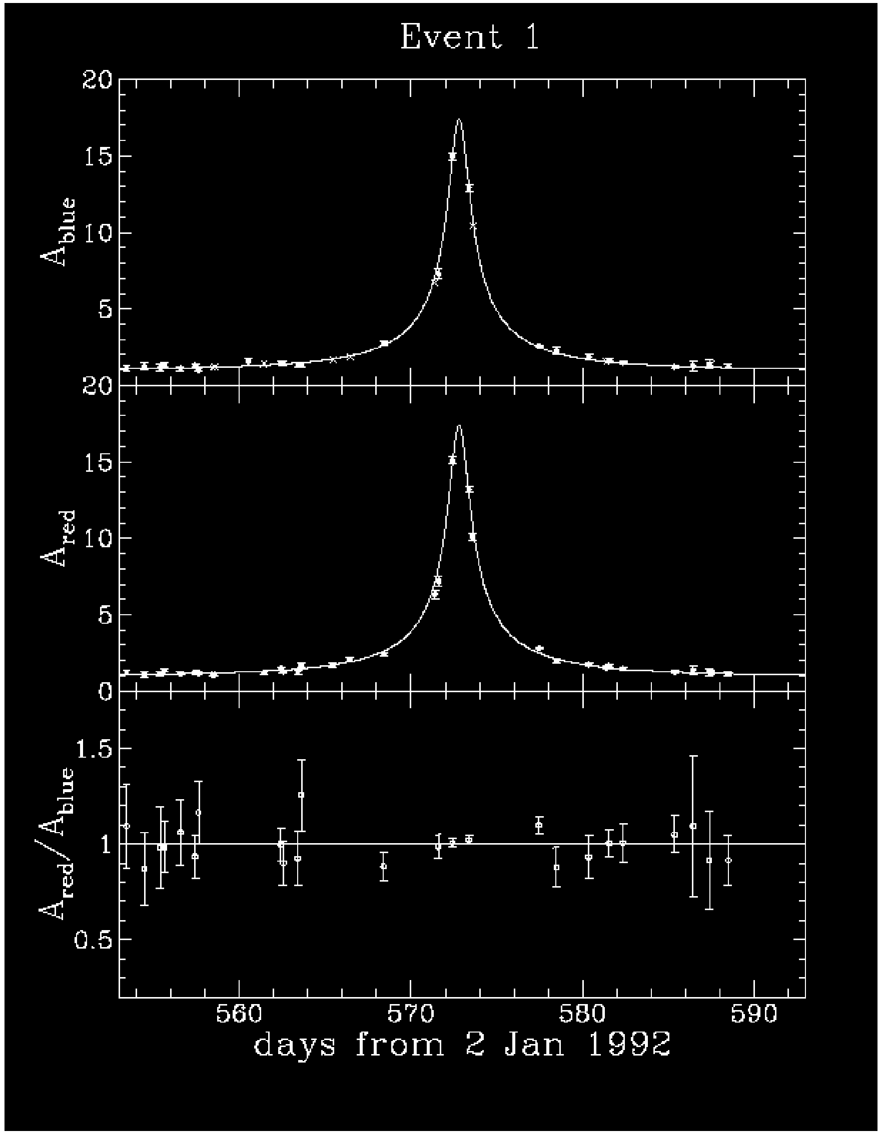


- Lots of stars are variable
- However, these stars will get brighter/dimmer equally at all wavelengths
- And it will follow curve predicted by theory

- Realistically, watch thousands of stars in small area
  - Bulge is a good place to look for them

How Many MACHOS Are There?

- We can study many background stars by looking at the bulge, or at the Larger Magellenic Cloud (a small, nearby galaxy)
- Let a computer watch *many* stars and flag those that change brightness
- If they do, study them over time
- Compare multiple wavelengths (variable stars tend to change temperature)



Conclusions:

- MACHOS exist
- Mostly white dwarfs
- Substantial fraction of stars, but *not* the dark matter

What Is the Dark Matter? (2)

Dark matter candidates

- Cold hydrogen gas - Probably not
- White dwarfs
- Neutron stars
- Black Holes
- Brown dwarfs/"Jupiters"

  } Probably not
- Invisible massive particles

- We already argued against gas

- MACHOS seem to be ruled out
  - A caveat – very small or very large black holes might still work

- Invisible massive particles seem to work
  - Neutrinos are particles that we know exist and have mass
    - But they probably won't work
  - No other known particles work
  - But many speculative theories contain such particles

Orbits of Disk Stars

Conservation of Angular Momentum

What governs stellar orbits of disk stars:

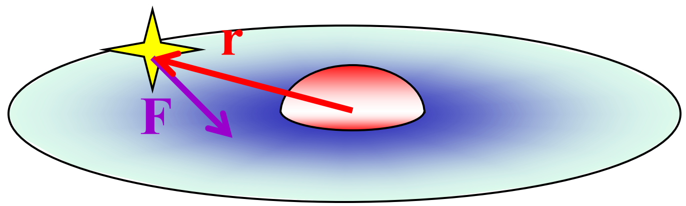
- For the most part, we can treat all the "other" stars as being uniformly spread out
    - Don't worry about effects of individual other stars

What conservation laws on a particular star can we use to figure out the motion?

- Momentum of star is not conserved – there are forces on it
- Energy conversation helps – but I won't use this
- Angular momentum conservation?

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \frac{1}{m}\mathbf{p} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$



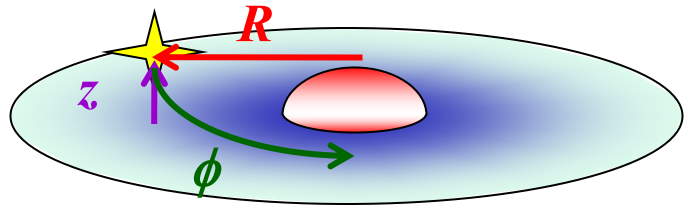
- The cross product $\mathbf{r} \times \mathbf{F}$ will <u>not</u> generally vanish
- However, the cross product *will* always be perpendicular to the vertical direction
    - Call this the z –direction
- This *component* of the motion will be conserved
- The combination Rv_ϕ is conserved

$$L_z = (\mathbf{r} \times \mathbf{p})_z = mRv_\phi$$

Three Types of Motion

$$Rv_\phi = \text{constant}$$

- We will be using cylindrical coordinates:
  - R – the distance from the z-axis, v_r the corresponding velocity
  - z – the vertical distance from the plane, v_z the corresponding velocity
  - ϕ – the angle around, v_ϕ the corresponding velocity



- There will potentially be three kinds of motion
- These will have associated with them three angular frequencies
  - κ – angular rate at which it wanders in and out
  - ν – angular rate at which it bobs up and down
  - Ω – angular rate at which it goes around

Angular motion – the easiest to understand

V^2 known

$$\Omega = \frac{V_0}{R_0}$$

- The star goes *approximately* in a circle
- Assume we know the angular velocity for circular orbits: $v_\theta\big|_{R_0} = V\big|_{R_0} = V_0$
- Assume that at some radius R_0 this is exactly v_θ
- Approximate angular velocity and angular period is:
- At all other radii, we must have: $Rv_\phi = R_0 V_0$

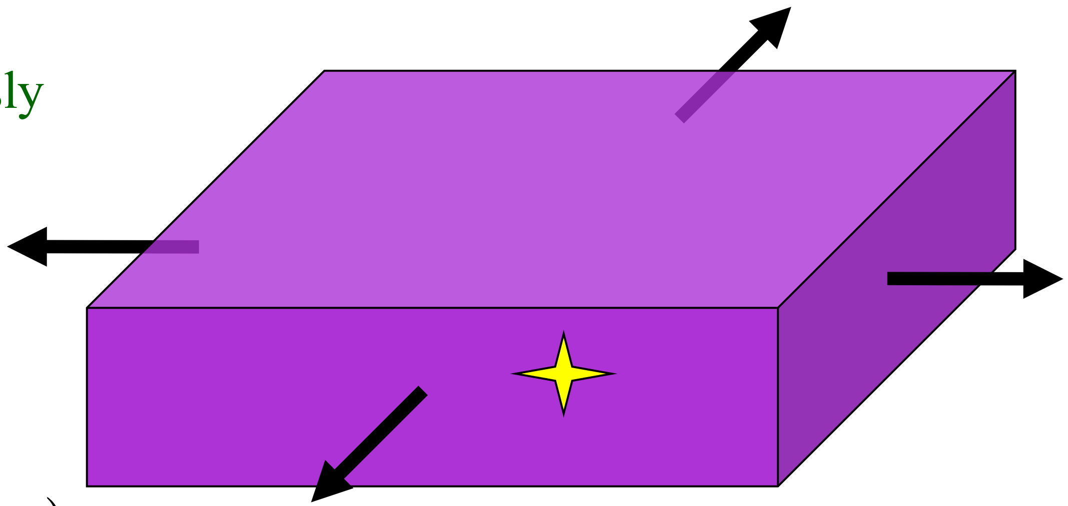
$$v_\phi = \frac{R_0 V_0}{R}$$

$$T_\phi = \frac{2\pi}{\Omega}$$

Up and Down Motion

In the vertical direction, there will be small motions

- The star should only be moving a small amount, z small
- Locally, the disk looks much like a uniform slab
- We found the gravitational acceleration previously



$$\mathbf{g}(z) = -4\pi G\rho_0 z\hat{\mathbf{z}} \qquad \frac{d^2z}{dt^2} = -4\pi G\rho_0 z$$

- This looks like Hooke's Law
  - Simple harmonic motion

$$\nu = \sqrt{4\pi G\rho_0} \qquad \frac{d^2z}{dt^2} = -\nu^2 z \qquad z(t) = z_0 \sin(\nu t)$$

- Star bounces up and down

$$T_z = \frac{2\pi}{\nu}$$

In and Out Motion: Epicycles (1)

In the radial direction, star moves in and out somewhat

$$v_\phi = \frac{R_0 V_0}{R}$$

- I will work in a frame that is rotating around with the galaxy
  - In this frame, there will apparently be a centrifugal force

$$F_c = \frac{mv_\phi^2}{R}$$

- If it were in a perfectly circular orbit, force would cancel gravitational force, so

$$F_g = -\frac{mV^2}{R}$$

- Effective force is the sum of these two

$$F_r = F_c + F_g = \frac{mv_\phi^2}{R} - \frac{mV^2}{R} = m\frac{d^2R}{dt^2}$$

- This will equal mass times radial acceleration

- We are only interested in near circular orbits, so $R \approx R_0$ circular and $V \approx V_0$
- Expand, keep only leading order term

$$\frac{d^2R}{dt^2} = \frac{R_0^2 V_0^2}{R^3} - \frac{V_0^2}{R} + \frac{V_0^2}{R} - \frac{V^2}{R} = \frac{V_0^2\left(R_0^2 - R^2\right)}{R^3} + \frac{V_0^2 - V^2}{R}$$

In and Out Motion: Epicycles (2)

$$\frac{d^2R}{dt^2} = \frac{V_0^2\left(R_0^2 - R^2\right)}{R^3} + \frac{\left(V_0^2 - V^2\right)}{R}$$

- Taylor expand V^2 around R_0, then substitute:

$$V^2 \approx V_0^2 + \left(R - R_0\right)\frac{d}{dR}V^2\Big|_{R_0}$$

$$\frac{d^2R}{dt^2} = -\left(R - R_0\right)\frac{V_0^2\left(R + R_0\right)}{R^3} - \left(R - R_0\right)\frac{1}{R}\frac{d}{dR}V^2\Big|_{R_0}$$

- Keep only leading order

$$\frac{d^2R}{dt^2} = -\left(R - R_0\right)\left[\frac{2V_0^2}{R_0^2} + \frac{1}{R_0}\frac{d}{dR}V^2\Big|_{R_0}\right]$$

$$\kappa^2 = \frac{2V_0^2}{R_0^2} + \frac{1}{R_0}\frac{d}{dR}V^2\Big|_{R_0}$$

- This is yet another Harmonic oscillator

$$\frac{d^2R}{dt^2} = -\kappa^2\left(R - R_0\right)$$

$$R(t) = R_0 + (\Delta R)\sin(\kappa t)$$

$$T_r = \frac{2\pi}{\kappa}$$

Example: flat rotation curves:

$$\kappa^2 = \frac{2V_0^2}{r_0^2} = 2\Omega^2 \qquad \kappa = \sqrt{2}\Omega \qquad T_r = \frac{T_\phi}{\sqrt{2}}$$

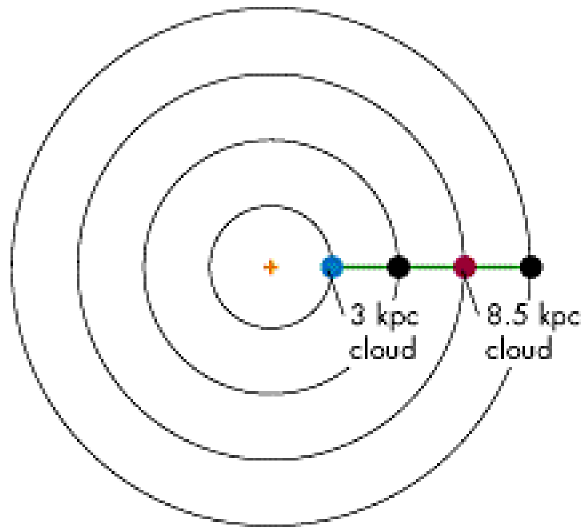
Spiral Arms: What Causes Them?

- Spiral Arms vary between galaxies
  - Most spiral galaxies have two arms
  - Some have three or four
  - Some have "partial" spiral arms

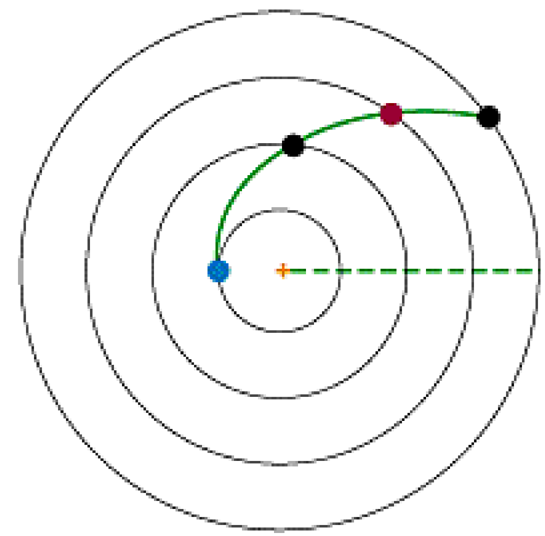
- The "winding" nature of them is a bit tricky
  - Not just simple winding!

Simple Winding: The Wrong Theory

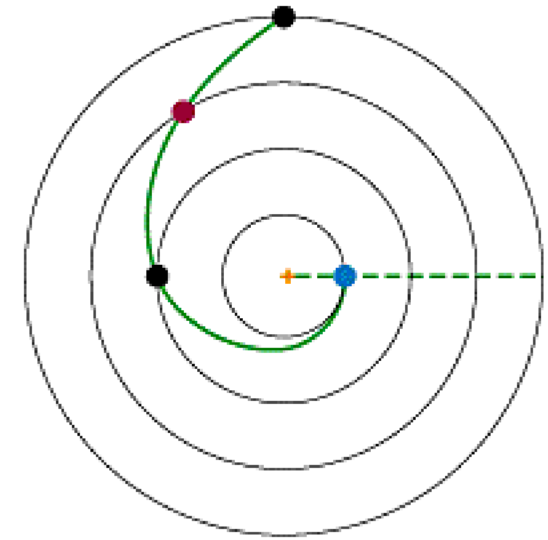
- In one cycle, spiral arms would end up completely wound up
- There have been 20 or so circuits since the beginning
- Therefore, it's not this simple
- It cannot be the *same stars* that inhabit the spiral arm on each cycle
- Different stars, clouds and gas inhabit it in each cycle



A A string of gas clouds lines up radially



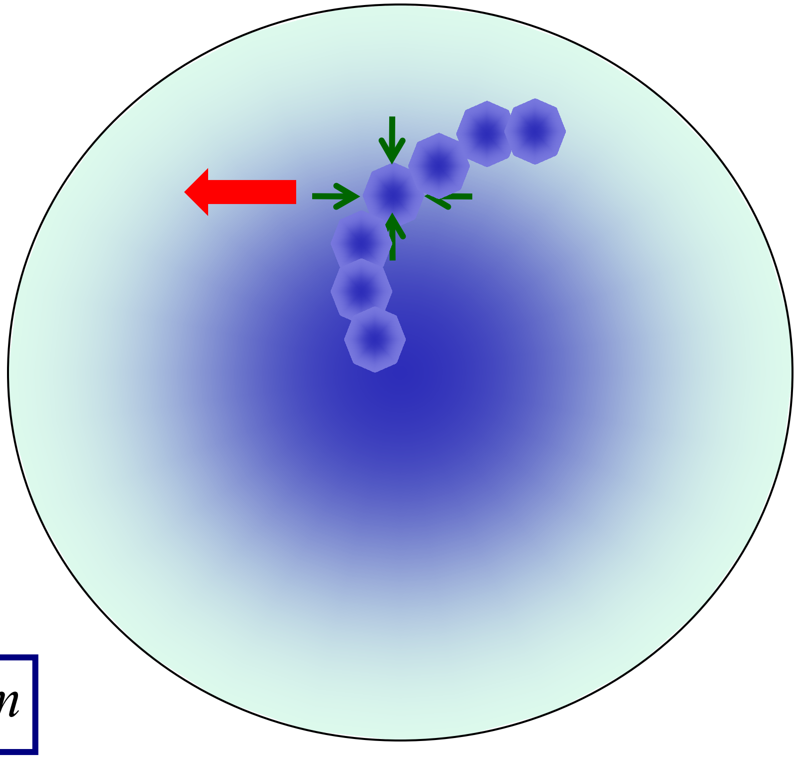
B The 3 kpc cloud completes half of a revolution in the time the 8.5 kpc cloud completes $\frac{1}{6}$ of a revolution



C The 3 kpc cloud will pass the 8.5 kpc cloud in little more than one orbit

Density Waves: The Idea

- Suppose a region in a rotating spiral galaxy has higher density "clump"
  - Due to random fluctuations, nearby galaxies, etc.
- It attracts gas from in front, from behind, from inside, from outside
- But they rotate at different rates, so the "clump" gets spread out
- Which causes still more clumps to form
- The pattern angular frequency Ω_{gp} will be a little different than the rotation rate Ω
  - Because it is spreading in both directions
- How widespread the pattern can be depends on how many arms *m* you need
- Can show that pattern only works if Ω_{gp} differs from Ω by at most κ/m.



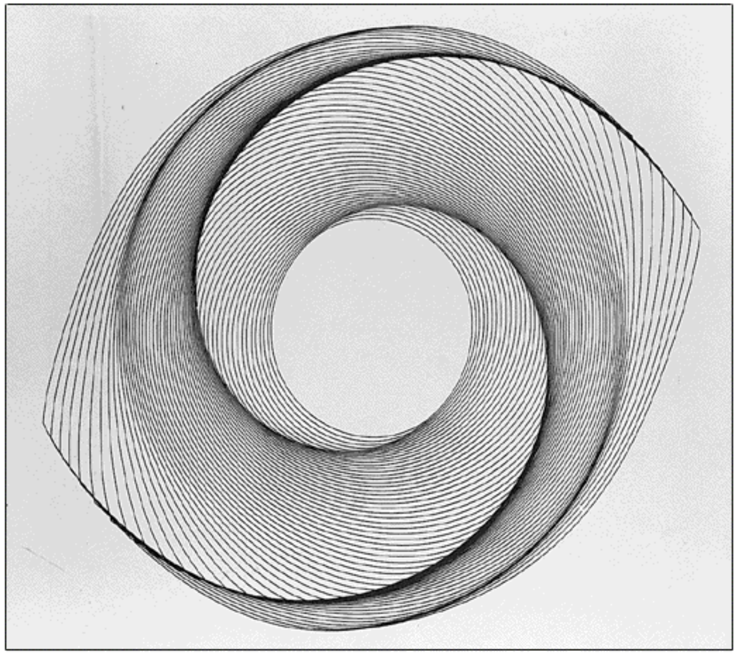
$$\Omega - \kappa/m < \Omega_{gp} < \Omega + \kappa/m$$

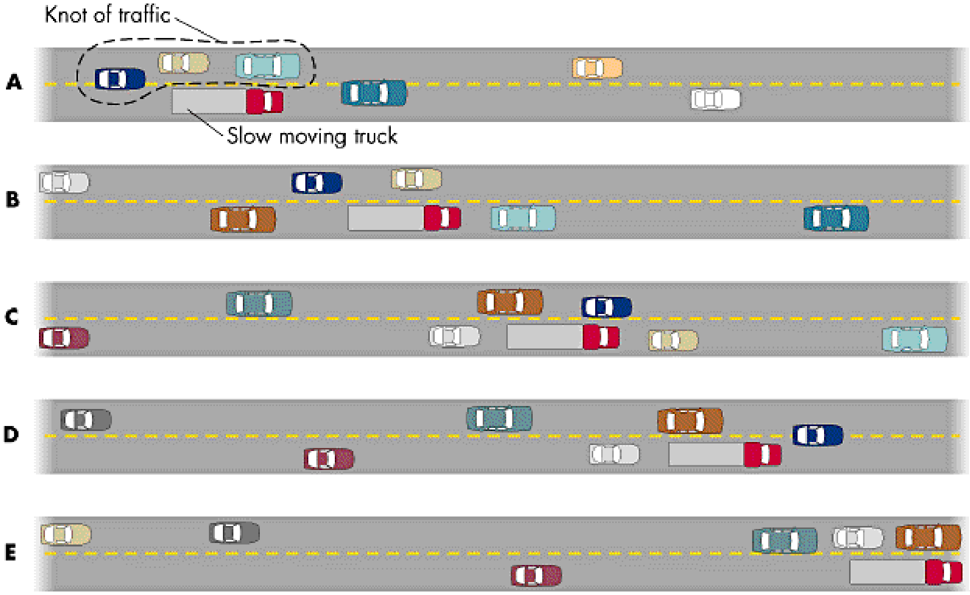
The Spiral Arms: What We're Seeing

- The "clumping" works best for objects with nearly perfect circular orbits to start with

- Works best for cool gas
  - Molecular clouds – almost in perfect circular orbits
- These regions are where the young stars will form
- Young stars (the brightest) mark out the spiral arms

- Once stars are born, they typically "fall out" of the spiral arms
  - The spiral arms are *not* made of particular stars – they change over time





Shapes of Clusters

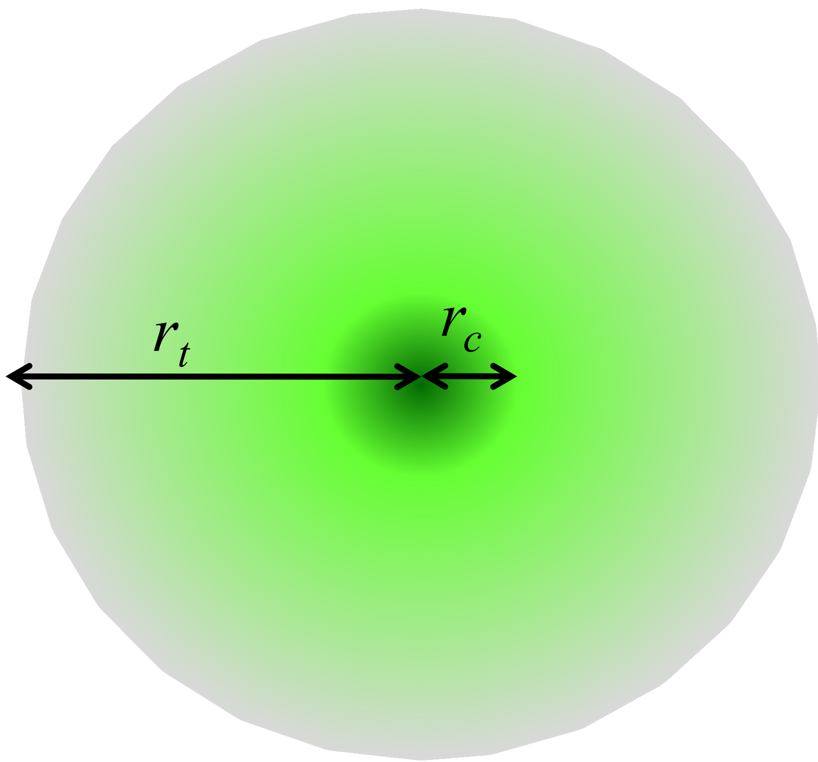
Core and Tidal Radius

Shape of a cluster (especially globular clusters)

- Typically roughly spherical

- Dense inner region
  - Core radius r_c

- Sharp dropoff at large radius
  - Tidal disruption radius r_t
  - Region where other gravitational objects have stripped stars away



Conservation Laws with Clusters

- Clusters (especially globular clusters) *most* of the time have relatively little interaction with other objects
  - Conservation laws should hold within the cluster

- Though they have net momentum, we can ignore that
  - Work in center of mass frame of the cluster

- They usually were formed with little or no net angular momentum
  - Generally, this will just be conserved, so they stay that way

- Over the course of approximately one orbit, potential energy ← → kinetic energy

- Therefore, over time, the system will *virialize*

$$2E_K + E_P = 0$$

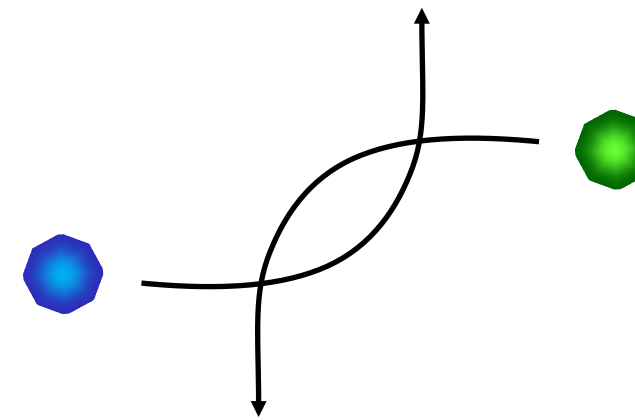
They then evolve due to two types of effects

- Close encounters of pairs of stars in the cluster

- Interaction of passing stars or other mass sources

Close Encounters of Pairs of Stars

- When two stars pass near each other, they will alter each others' orbits
- This changes each star's momentum and energy, but not the total
- Over time, this allows transfer of energy between all the stars



- System ends up in a sort of thermal type distribution
  - Don't think of this dynamical "temperature" T_D too literally
  - Does <u>not</u> correspond to the temperature of the stars themselves

$$P(E) \propto e^{-E/k_B T_d}$$

- Recall, energy is kinetic plus potential energy
- Note that *both* terms in the energy are proportional to mass
- The probability distribution prefers lower energy states
  - This effect has the most effect on high mass stars

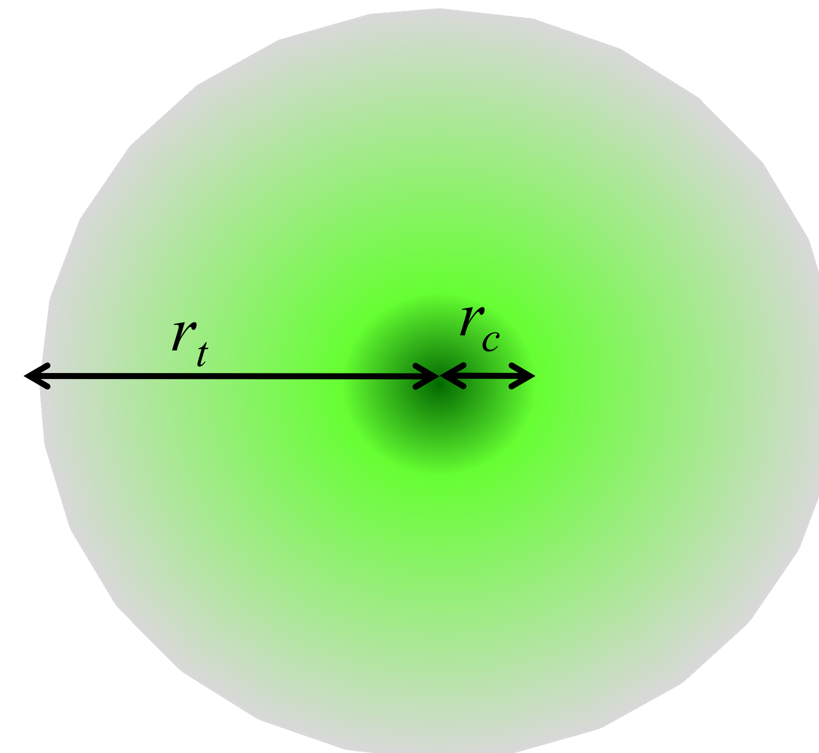
$$E = E_K + E_P = \tfrac{1}{2} m v^2 + m\Phi$$

- Therefore:
  - High mass stars tend to move at lower velocities
  - High mass stars tend to "fall" to the center of the cluster

Evolution of Cluster Shapes

Due to interactions between stars within the cluster:

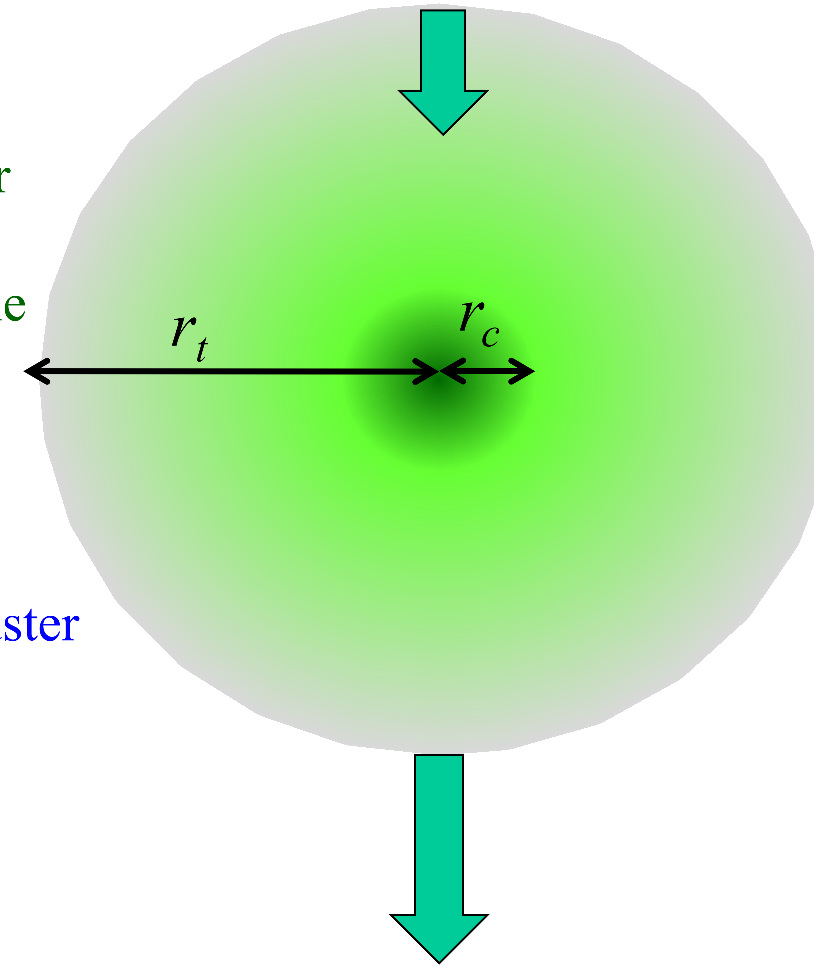
- Because the distribution depends only on energy, and this is the same in all directions, the cluster ends up as a sphere
- The more massive stars are gradually moving towards the center, and slowing down
- Less massive stars drift towards the edge
- Over time, the core radius r_c shrinks smaller and smaller
- Eventually r_c shrinks to zero
  - Many globular clusters seem to have already reached this stage
- The outer layer *should*, over time, expand and slowly evaporate off
  - But that's not what we see



Effects of Passing Mass on a Cluster (1)

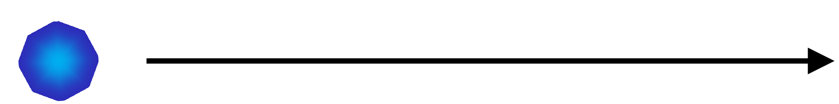
What effect does a passing mass have on a cluster?

- Typically, the passing mass is moving quickly past the cluster
- As it goes by, it pulls on the stars
- Net effect: The entire cluster accelerates in the direction of the passing star
  - But not evenly!



- We don't care about the net motion of the cluster
- But we do care about what effect is has on the stars in the cluster
- It pulls most strongly on the part of the cluster near it
- It pulls more weakly on the far part of the cluster
- It pulls diagonally on the parts to the side

- This means it is adding internal kinetic energy to the cluster



Effects of Passing Mass on a Cluster (2)

$$E = E_K + E_P$$

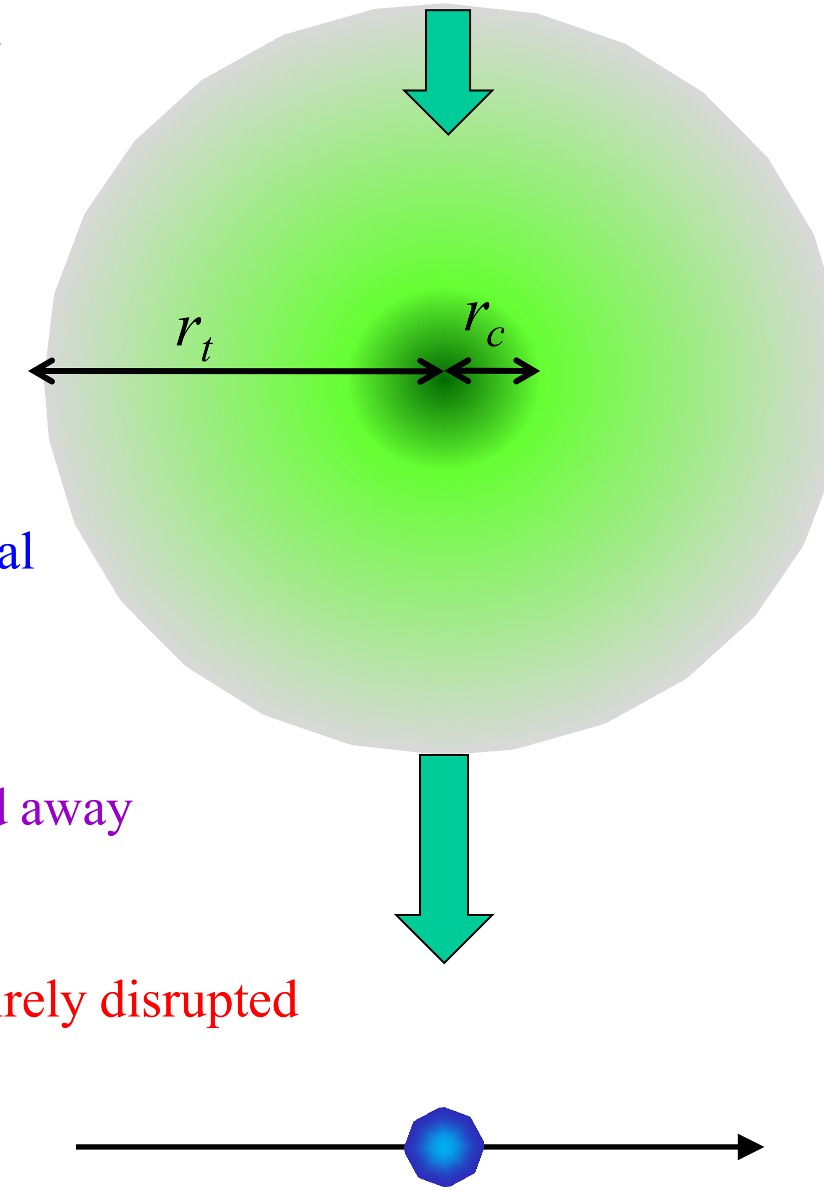
$$E_k = \tfrac{1}{2}\left|E_p\right|$$

- The initial energy was negative
  - Can be easily seen from the total energy and the virial theorem

- The total energy has increased
- Over one dynamical time scale, this energy will get distributed between the kinetic and potential energy according to the virial theorem
- The energy gets redistributed so that the kinetic and potential energy is shared

- The cluster gets larger and more loosely bound
- Stars near the outer edge eventually get completely stripped away
- This causes there to be a relatively sharp outer boundary

- If the cluster has insufficient mass, it will eventually be entirely disrupted



CAUTION!

- Many of the details about how galaxies form structure are not well understood
  - Much of our understanding comes from computer simulations, without detailed theories
- If an expert told you everything they knew, some of it would be wrong
- I am not an expert, and hence some of what I am going to tell you is *probably* wrong
- Take my comments as probably generally right, but probably wrong in details
- And it will doubtless need revision over time

Shapes of Galaxies

Conservation Laws in Disks

- The stars in a disk are formed from gas and dust in the disk
  - We need to understand these objects to understand the stars in them
- Although a galaxy is moving, we are once again not interested in the *net* motion of the galaxy
  - Work in the center of mass frame of the galaxy

- Unlike stars, gas clouds are *huge* and frequently undergo collisions
- These collisions heat the gas
- The gas then starts to radiate the heat, which leaves the disk
- So we have Kinetic Energy → Heat → Radiation → Lost

- Effectively, energy is *not* conserved in the gas of the disk

- However, radiation carries off very little momentum
- And therefore, very little angular momentum

- Angular momentum *is* conserved in the disk

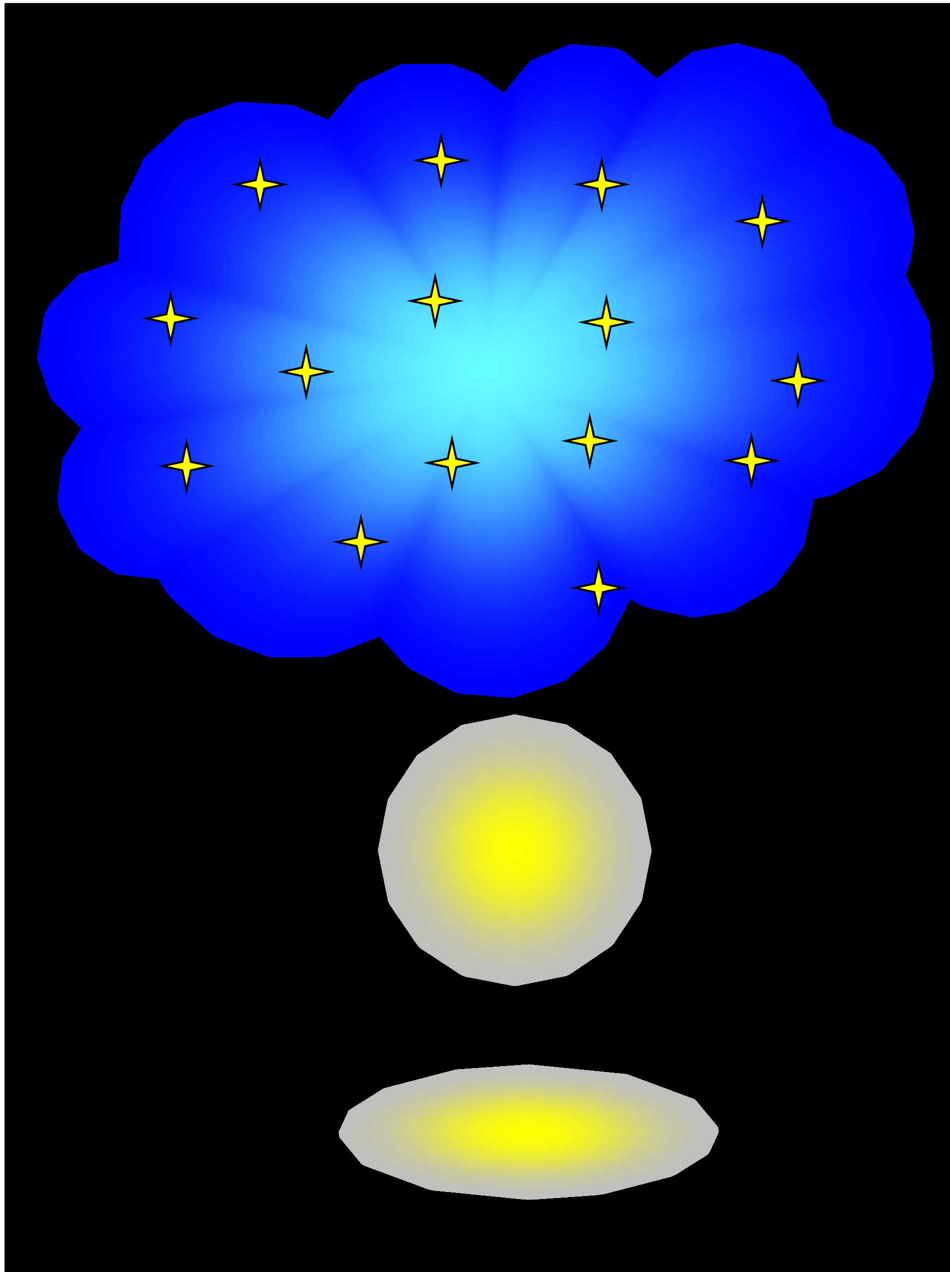
Early Star Formation In Galaxies

- All stars come initially from clouds of gas
- Initially, the clouds of gas can be any shape
- Stars form throughout the cloud
  - Initially probably having little motion

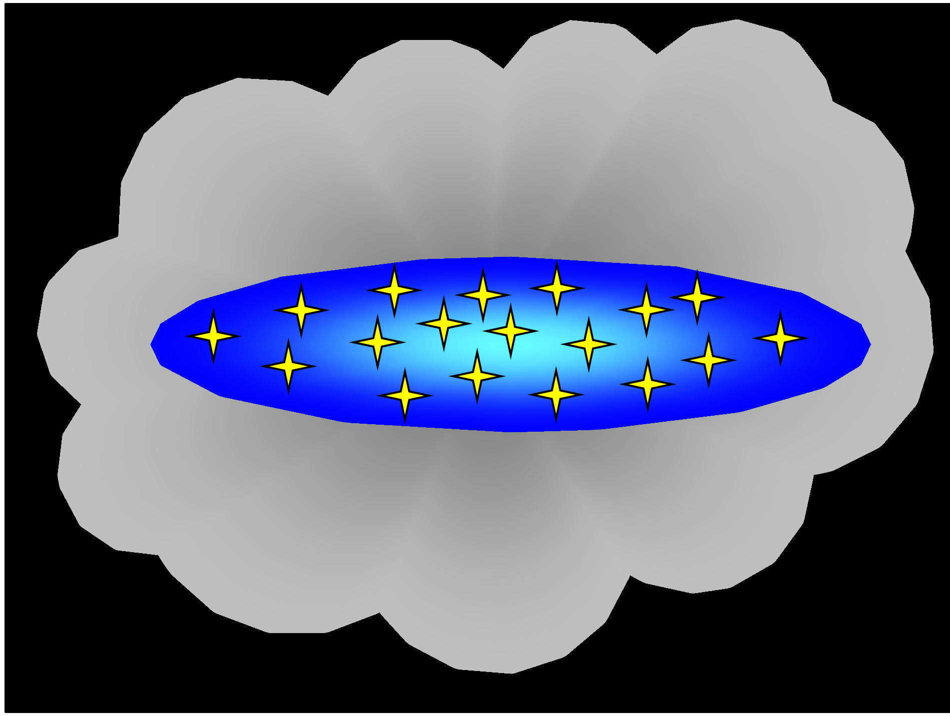
- Gravity pulls the stars towards the center
  - Converting gravitational energy to kinetic
- Ultimately, the system virializes

- If there is no net angular momentum, it will forma sphere
  - With stars having random motion in it

- If there is some net angular momentum, it will form an oblate sphere
  - Stars having some random motion
  - But more going in direction of rotation than counter to it
- These early stars may be the bulge stars



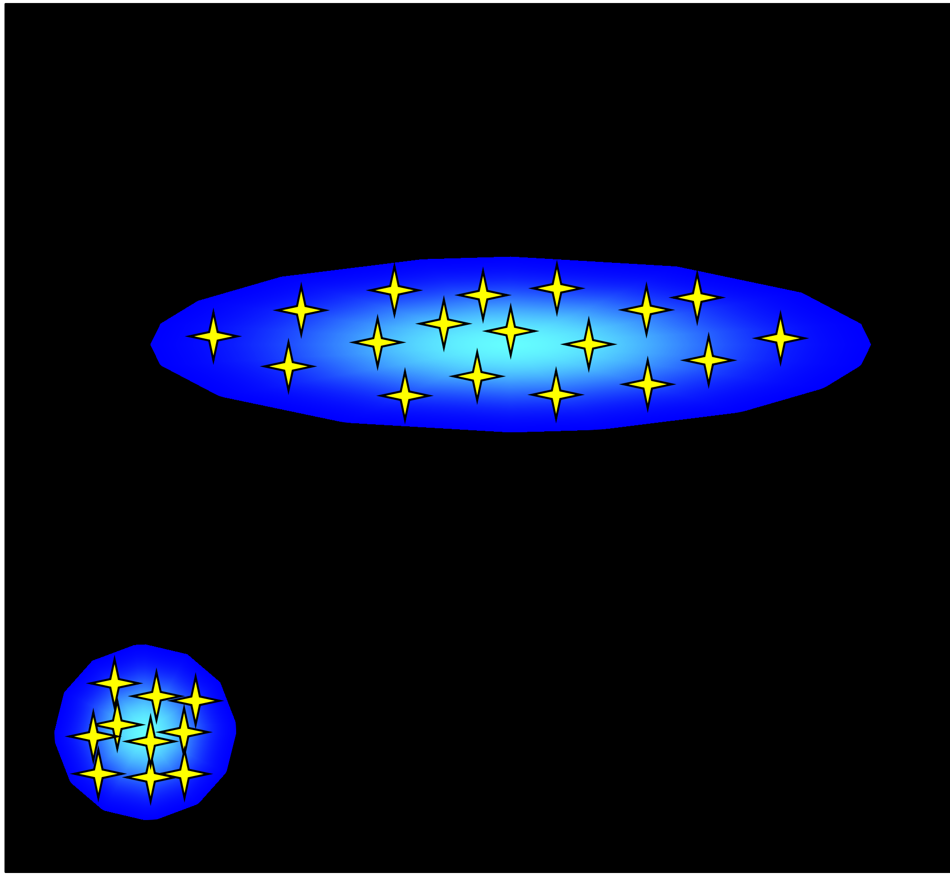
The Shapes of Disks



- The cloud of gas is also getting pulled together
- Because energy is not conserved in this cloud, it can shrink down a lot more
- It can't shrink to a point if it has angular momentum
- It will ultimately become a disk
- Only thing opposing it becoming infinitely thin is pressure
- The lowest temperature gas (molecular clouds) will make a very thin disk
- New stars will form in this disk
- Hence the youngest stars always form in the thinnest disk

Evolution of Shapes of Disks

- Passing galaxies and collisions with small galaxies will add kinetic energy to the remaining gas and stars
- This causes orbits to distort, no longer circular, and moving above and below the plane
- The gas ultimately loses this excess energy and goes back to being a disk
- But the disk of the stars thickens permanently
- The older parts of the disk will tend to be thicker than the younger parts



Instabilities in Disks and Bulges

- A perfectly symmetric ellipsoid should remain that way indefinitely
- But passing galaxies and other perturbations cause distortions
- In the bulge, there is an instability that makes the oblate ellipsoid become more elongated (cigar shaped)
- In the disk, there are instabilities that cause spiral arms to form
- Much of this information comes from computer simulations