Name ___________________

Solutions to Final Exam
March 30, 2010

This test consists of three parts. For the first part, you may write your answers directly on the exam, if you wish. For the other parts, use separate sheets of paper. Useful equations can be found at the start of part 3. The total test is worth 300 points.

Part I: Multiple Choice [60 points]
For each question, choose the best answer (2 points each)

1. The fact that spiral galaxies contain dark matter that is *not* concentrated in the center is evident when you
   A) Realize the extremities of the galaxies are much darker than they should be
   B) Plot rotational velocities as a function of distance, and discover they do not fall off with distance
   C) Plot rotational velocities as a function of distance, and discover they fall off too fast with distance
   D) Study the temperature of X-rays in the disk of the galaxy
   E) Realize how little mass is in the black hole in the nucleus

2. Which of the following is *not* a likely outcome of a direct collision between galaxies?
   A) The gas contained in the galaxy can become very hot
   B) A sudden burst of star formation
   C) Collision and merger of significant numbers of individual stars
   D) One of the galaxies may become an active galaxy
   E) Actually, all of these *are* common outcomes of galaxy collisions

3. Which of the following might be the current value of Ω_m, the fraction of critical density the form of matter of any kind?
   A) 6×10^{-5} B) 0.046 **C) 0.27** D) 0.73 E) 1.00

4. The most important thing about a standard candle used for measuring distance is that it have a consistent (same for all members):
   A) Luminosity B) Temperature C) Distance D) Brightness E) Radius

5. Which formula describes how the scale factor *a* behaves during inflation?
   A) $a = \text{constant}$ **B)** $a \propto e^{Ht}$ C) $a \propto t^{1/2}$ D) $a \propto t^{2/3}$ E) $a \propto t^{2}$

6. To leading order, the Sun moves in a circle around the center of the galaxy. To greater accuracy, what else does it do?
   - A) The circle actually is growing slowly larger, so over time the Sun will move farther from the center of the galaxy
   - B) The Sun also bobs up and down compared to the plane of the galaxy, but it does not move radially inwards nor outwards
   - C) The Sun also moves radially inwards and outwards, but it does not bob up and down
   - **D) The Sun *both* bobs up and down *and* moves radially inwards and outwards**
   - E) None of the above is correct

7. As viewed from Earth, (almost) all galaxies seem to be moving away from us with a velocity proportional to distance. What would one see if you viewed the universe from some distant galaxy?
   - A) You would see all galaxies moving past you in some direction
   - B) You would see all galaxies moving towards you
   - C) You would see all galaxies moving away from you, but it would not follow Hubble's Law
   - D) You would see all galaxies moving away from you, and it would obey Hubble's Law, but with a different Hubble's Constant
   - **E) You would see all galaxies moving away from you, and it would obey Hubble's Law, and with the same Hubble's Constant**

8. In addition to ^{1}H, primordial nucleosynthesis also produced
   - A) Nothing else
   - B) ^{4}He, but no ^{2}H, ^{3}He, ^{7}Li, nor ^{12}C
   - C) ^{4}He, ^{2}H, and ^{3}He, but no ^{7}Li nor ^{12}C
   - **D) ^{4}He, ^{2}H, ^{3}He and ^{7}Li but not ^{12}C**
   - E) ^{4}He, ^{2}H, ^{3}He, ^{7}Li and ^{12}C

9. In which portion of our galaxy is the Sun located?
   A) Disk B) Halo C) Nucleus D) Bulge E) Globular Cluster

10. Which of the following would *not* be a clue that a nearby star is actually a halo star, rather than a member of the galactic disk?
    - A) Low metallicity Z
    - **B) A member of an open cluster**
    - C) Large vertical motion
    - D) Non-circular orbit, perhaps even going backwards
    - E) Actually, all of these *are* clues that it is a halo star

11. When attempting to get an idea of the age of a cluster of stars, you can get a pretty good estimate by studying a Hertzsprung-Russell diagram and noticing
    A) The greatest luminosity of stars in the cluster
    B) The fraction of stars that have moved into the giant stages
    C) The turn off point where the cluster deviates from the Main Sequence
    D) The coolest stars
    E) The number of stars that are white dwarfs

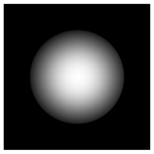
12. How many *large* galaxies (comparable or larger than our own galaxy) are there in our cluster? Include our own galaxy.
    A) 1 **B) 2** C) 3 D) 4-5 E) More than 5

13. The best method for estimating the mass of a galaxy cluster is
    A) Add the mass of each component galaxy to get a total
    B) Measure the total amount of X-rays, and from this deduce the total mass
    C) Measure the amount of gravitational lensing acting on some bright object(s) behind the cluster
    D) Measure the velocity at which a satellite galaxy orbits the collection
    E) Just make up an answer; no one will ever know

14. Which of the following is *not*, as far as we can tell, understandable in terms of conventional physics and cosmology (including inflation, for example), but might be best explained in terms of the anthropic principle
    A) Why the dark energy/vacuum energy density is so low
    B) Why the universe looks almost exactly flat
    C) Why the universe is apparently the same in all directions
    D) Why there is cosmic microwave background radiation
    E) Why there are more baryons than anti-baryons in the universe

15. The velocity of a star towards or away from us is measured by measuring
    A) How fast its apparent radius changes as viewed by us
    B) Its color – shifted towards blue if moving towards us, red if away
    C) The peak wavelength of its spectrum
    D) The Doppler shift of its spectral lines
    E) The change in brightness over time

16. What would be the likely galaxy classification shown at right?
    A) Sd B) S0 C) E7 **D) E0** E) Irr



17. Which of the following is problems with neutrinos as the dark matter, hot dark matter?

I. Neutrino masses are so big they would be too large a contribution to Ω
II. Neutrinos tend to “free stream” away at high speeds, wiping out any structure
III. Neutrinos are fermions, and it’s tough to get them to “fit” in galaxy without violating the Pauli Exclusion Principle
A) I only is true
B) II only is true
C) III only is true
D) I, II, and III are all true
E) II and III are true, but not I

18. Which of the following distance methods is used to study the distance to some of our nearest neighbor stars?
A) Parallax
B) Cepheid Variables
C) Type Ia Supernovae
D) Hubble’s Law
E) Light Echo Method

19. On the largest scale, our universe looks most like
A) A sponge: Mostly empty space (voids) with relatively thin regions of stuff between them
B) A soft drink: Mostly uniform but with little “bubbles” of empty space every so often
C) A plate of spaghetti: Stuff arranged on long, thin structures, but fairly tightly packed
D) A hollow ball: Galaxies arranged on a thin outer structure with vast gaps between
E) A (Rutherford) atom: Most of the mass in the center, with the rest in little bunches circling the center

20. Recombination occurred at $z = 1091$. If you use that in the formula $t = \left(17.8\text{ Gyr}\right)\left(1+z\right)^{-3/2}$, you will get an age of 493,000 years, but the correct age is 380,000 years. The cause of this discrepancy is that:
A) $z = 1091$ requires the other formula
B) The universe is close to the radiation/matter dominated boundary, so neither formula works quite correctly
C) The red shift is so large that relativistic effects (which have not been included) must be taken into account
D) The process of recombination *itself* caused a shift in wavelengths, effectively shifting the value of z somewhat
E) The cause is unknown, but probably some error in Hubble’s constant

21. It is possible that dark energy *increases* as a small positive power of the scale factor *a*. If this is the case, what do we expect will be the ultimate fate of the universe?
   A) A whimper: everything will simply expand away exponentially forever
   B) The big crunch, with everything coming back together to a point
   C) Static, with everything coasting to a stop
   D) Quantum nucleation, with new universes spontaneously appearing from the old
   E) The big rip, with expansion so fast the universe will reach infinite size in finite time

22. If we go to very early times, like back to the time of electroweak symmetry breaking, what would have been the approximate value of Ω?
   A) Significantly less than one
   B) Significantly more than one
   C) Approximately 1, but we don't know if it was really close to 1
   D) 1.0000000000…; that is, one to many digits
   E) The value of Ω at this time is not known, even approximately

23. In class, I explained that it is likely that most of the cold dark matter that existed in the early universe was eliminated by processes such as (for example) $X\bar{X} \to e^+e^-$. According to this hypothesis, why wasn't *all* the cold dark matter eliminated this way?
   A) There is no longer sufficient energy to make electron/positron pairs
   B) The intermediary particles that catalyze this process have disappeared
   C) The *X*'s have dropped to such a low density they can no longer find each other
   D) This process can only work if supersymmetry is unbroken, but we have dropped below the supersymmetry scale
   E) Any electron/positron pairs created by this process will simply be attracted to each other and make more *X*'s

24. When we look at the cosmic microwave background, we see the universe at the time of:
   A) Baryon creation
   B) Primordial Nucleosynthesis
   C) First structure formation
   D) Recombination
   E) Inflation

25. Early on (say at $k_BT = 1$ MeV) there are plenty of protons and neutrons, but they don't *immediately* proceed to make heavier elements. How come?
   A) The temperatures are so high that ^{2}H, the first step, even if made, immediately gets blasted apart before it can get beyond this stage
   B) This is above the quark confinement temperature, and if you can't confine quarks, you certainly can't build nuclei
   C) The age of the universe is so short at this point that the protons and neutrons simply don't have time to find each other
   D) The universe is opaque (this is before recombination), so if they combined, the energy simply cannot escape as a photon
   E) Protons and neutrons have no net charge, and hence aren't attracted to each other

26. For a radio galaxy or a radio quasar, where is most of the radio power coming from?
   A) The nucleus of the galaxy, where the black hole is
   B) A pair of jets usually contained within the disk of the galaxy
   C) The entire disk of the galaxy
   D) A small region near the center, but not at the black hole
   E) Huge radio lobes that typically stick out of the galaxy, often perpendicular to it

27. In order from the largest contributor to the smallest to the mass density today, we would have
   A) Dark matter, dark energy, conventional matter
   B) Dark matter, conventional matter, dark energy
   C) Conventional matter, dark matter, dark energy
   D) Dark energy, conventional matter, dark matter
   E) Dark energy, dark matter, conventional matter

28. If we wished to know the distance to Venus, we could, in principle, simply bounce a radar off of it and measure the time it takes for the echo to return. The reason we don't do this is
   A) The signal would be too weak to see by the time it got there and back
   B) There are far more accurate ways to do the distance, such as parallax
   C) The speed of radar is too poorly measured to make this useful
   D) The atmosphere of Earth and/or Venus blocks radar
   E) Actually, this *is* how we measure distances within the solar system, such as the distance to Venus

29. According to our current understanding, how did structure form in the universe
   A) Bottom up: First the smallest structures (globular clusters), then larger, working up to the largest structures
   B) Top down: First the largest structures, then gradually working down to globular clusters
   C) Inside out: First moderate sized structures, then it worked both directions from there
   D) Outside in: First the largest and smallest structures, then it gradually worked its way to intermediate sizes
   E) Uniform: Structures on all sizes formed at about the same time

30. It is generally assumed that neutrinos are a bit colder than photons ($T_\nu = 0.714T$). What is the cause of this difference?
   A) Neutrinos transferred a small amount of their energy to nuclei during nucleosynthesis
   B) The photons got heated up slightly in the modern era by stars and so on
   C) Electrons and positrons annihilated, which increased the energy of the photons but not the neutrinos
   D) As the universe expanded, the neutrinos cooled more because they are fermions
   E) Neutrinos have very weak interactions, and though they *actually* have the same temperature, they therefore have *effectively* a somewhat lower temperature.

Part II: Short essays [120 points]

Write approximately a paragraph or two explaining, or otherwise follow the instructions, for each of the following (15 points each).

31. Ordering – Place the seven events at right in the correct order, from first to last.

The correct ordering is given in the table I have added.

Planck Era	Baryons Created
Baryons Created	Electroweak Scale
Electroweak Scale	First Structure Forms
Quark Confinement	Matter/Radiation Equality
Neutrinos Decouple	Neutrinos Decouple
Matter/Radiation Equality	Planck Era
First Structure Forms	Quark Confinement

32. At a temperature of about 0.7 MeV, the processes that turn protons into neutrons and vice versa "froze out," with one neutron for roughly every six protons. Yet the universe today contains nearly no free neutrons. Explain qualitatively what became of these neutrons. You should be able to explain, on the basis of this, the *approximate* fraction of one of the isotopes produced in the Big Bang.

At first, the temperature was so high that these neutrons could not bind with protons, but eventually, the temperature dropped enough that these neutrons almost all found protons to bind with and make deuterium. Further processes, typically adding another proton and neutron, then built the nuclei up to ^{4}He. To leading order, all the neutrons ended up in ^{4}He.

A naïve calculation allows one to figure out the mass fraction of ^{4}He. For every fourteen nucleons, there will be two neutrons, which combine with two protons to make one ^{4}He nucleus. This has a weight of about 4 u, so 4/14 of the total mass, or about 29% would end up as ^{4}He.

This computation is slightly wrong, because, in fact, a small fraction of the neutrons have decayed, resulting in a diminished fraction of ^{4}He. The actual mass fraction is very close to 25%.

33. In general, a stellar system might be a member of a galaxy cluster, a galaxy, a stellar cluster, and a galaxy supercluster. Reorder these four objects into the correct order, from smallest to largest. Then tell me the name of each of these that we belong to (hint: we aren't necessarily a member of all of these).

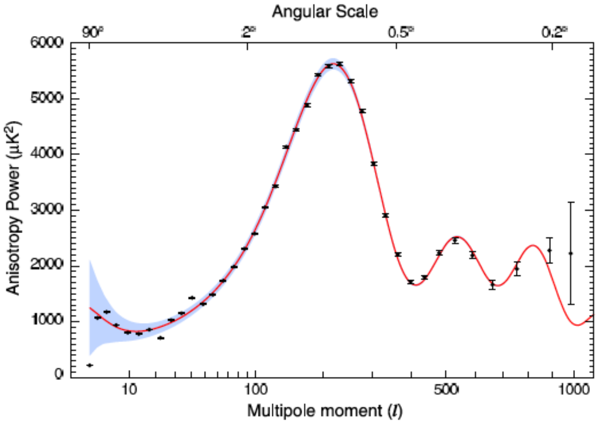
The correct ordering, as well as the solar system's place, is given in the table at right. The Sun is not a member of a stellar cluster.

Stellar Cluster	(none)
Galaxy	Milky Way
Galaxy Cluster	Local Group
Galaxy Supercluster	Virgo Supercluster

34. Explain qualitatively some of the successes of inflationary theory. You should have at least three.

Inflation correctly explains both the fact that the universe is flat or very nearly so (Ω = 1) as well as its homogeneity, the fact that disparate parts of the universe are at nearly identical temperatures (if you look at the temperature fluctuations) and galaxies are nearly uniformly distributed (if you look at galaxy distributions).

Inflation also explains the origin of those density fluctuations. According to inflation, there will be nearly scale invariant fluctuations from the early universe, though there is slightly more “power” at smaller scales. If we take into account how the fluctuations would have evolved in an expanding universe as we move from era to era in the early universe, we end up with temperature fluctuations very similar to what we see today. In particular, inflation predicts Gaussian fluctuations with the correct scale dependence.



35. If you plot the size of fluctuations in the microwave background as a function of *l* (or angle), the position of the first peak tells us approximately the value of what important cosmological parameter? Why does this parameter affect the position of this peak?

We are looking at a particular scale, the scale corresponding to the largest wave that doesn’t oscillate as an acoustic wave in the early universe. The angular scale of this wave depends on many things, but importantly, it depends on the shape of the universe as a whole. If the universe is flat, then the angular size is just the physical size divided by the distance. If the universe is closed, then because of the curvature of the universe, light rays will be bent inwards, effectively magnifying this scale and making its angular size bigger. If the universe is open, then the curvature of the universe as a whole demagnifies this scale, making it look smaller. Hence the angular size (which is equivalent to *l*) tells you a lot about Ω. At present, this is the most important input that leads to the conclusion that Ω=1.

36. Name four or more significant differences between a typical spiral or barred spiral galaxy like our own and a typical elliptical galaxy. One of them can be shape (*i.e.*, explain how their shapes differ).

The most obvious difference is appearance: Spirals have a spiral structure, which makes them easy to distinguish from elliptical, which are round or elliptical. Spirals have significant clouds of gas and dust, while ellipticals have little. Spirals contain a mix of old and young stars; ellipticals are almost all old stars. Spirals show signs of orderly rotation; stars in ellipticals tend to orbit in much more random orbits.

37. Explain qualitatively how Cepheid Variable stars can be used to measure the distance to relatively nearby galaxies, such as the Andromeda Galaxy.

Cepheid variable stars are stars that pulsate. Their rate of pulsation bears a simple relation to the maximum luminosity. Hence, if you measure the pulsation period (which you can) you can also estimate the luminosity. Since you know the brightness, or how bright an objects looks, you can combine this information to determine the actual distance, with the help of either $L = 4\pi d^2 b$ or the equivalent formula $d = 10^{1+(m-M)/5}$.

38. Arrange in the correct order, from longest wavelength to shortest, the seven categories of electromagnetic radiation at right. Also, mark which end has the highest energy for a single photon, and which has the lowest energy.

Radio	Gamma Rays
Microwaves	Infrared
Infrared	Microwaves
Visible	Radio
Ultraviolet	Ultraviolet
X-rays	Visible
Gamma Rays	X-rays

I've added a second table at near right with the correct ordering. The top has the lowest energy, and the bottom has the highest

Part III: Calculation: [120 points]

For each of the following problems, give the answer, explaining your work. The value of each portion appears in square brackets. Some possibly useful equations appear below. [20 points each]

Units

$$1\ \text{AU} = 1.496\times10^{11}\ \text{m}$$
$$1\ \text{pc} = 3.086\times10^{16}\ \text{m}$$
$$1\ \text{eV} = 1.602\times10^{-19}\ \text{J}$$
$$1\ \text{y} = 3.155\times10^{7}\ \text{s}$$
$$R_\odot = 6.955\times10^{8}\ \text{m}$$
$$M_\odot = 1.989\times10^{30}\ \text{kg}$$
$$L_\odot = 3.839\times10^{26}\ \text{W}$$
$$T_\odot = 5777\ \text{K}$$

Friedmann Equation

$$\frac{\dot{a}^2}{a^2} = \tfrac{8}{3}\pi G\rho - \frac{kc^2}{a^2}$$

Physical Constant

$$k_B = 1.381\times10^{-23}\ \text{J/K}$$
$$k_B = 8.671\times10^{-5}\ \text{eV/K}$$
$$\hbar = 1.055\times10^{-34}\ \text{J}\cdot\text{s}$$
$$\hbar = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$$
$$\hbar c = 1.973\times10^{-7}\ \text{eV}\cdot\text{m}$$
$$G = 6.673\times10^{-11}\text{m}^3/\text{kg/s}^2$$

Astronomical Constants

$$H_0 = 70.4\ \text{km/s/Mpc}$$
$$T_0 = 2.725\ \text{K}$$

Omega

$$\Omega \equiv \tfrac{8}{3}\pi G\rho / H^2$$

Radiation Density

$$\rho = g_{\text{eff}} \frac{\pi^2 (k_B T)^4}{30(\hbar c)^3 c^2}$$
$$g_{\text{eff}} = 3.36\ \text{(now)}$$

Time-Temperature Relations

$$t = \frac{17.8\ \text{Gyr}}{(1+z)^{3/2}}$$
$$t = \frac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_B T}\right)^2$$
$$z_{\text{eq}} \approx 3230$$

39. At right is a list of all elementary particles lighter than 500 MeV/c^2. The mass is given in MeV/c^2, and the number of spin states are listed as g. At some point in the universe, the temperature was $k_B T = 80\ \text{MeV}$.

Particles

Name	Mass	Spin	g
Photon	0	1	2
Neutrinos	~0	½	6
Electron	0.511	½	4
Muon	105.7	½	4
Pions	135-140	0	3
Kaons	495-500	0	4

(a) [2] Was the universe dominated by matter, radiation, or something else at this temperature?

At temperatures above about 1 eV, the universe is radiation dominated.

(b) [3] What would be an approximate average energy for a particle at this temperature? Which particles were probably abundant at this temperature?

The average energy of a particle is $3k_B T = 240\ \text{MeV}$. Particles that are lighter than this will tend to exist; particles heavier than this will be pretty much irrelevant.

(c) [7] Which particles are bosons and which are fermions? What is g_{eff} at this time?

Particles with integer spin are bosons, so the photon, pions, and kaons. Particles with half integer spins are fermions, so the neutrinos, electron and muon are all fermions. The kaons are not present, because they are too heavy, but the rest are. We then get g_{eff} using

$$g_{\text{eff}} = g_B + \tfrac{7}{8} g_F = (2+3) + \tfrac{7}{8} \cdot (6+4+4) = 5 + 12.25 = 17.25$$

(d) [8] Estimate the age of the universe at this time.

We now simply use the formula for the age during the radiation dominated era:

$$t = \frac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}} \left(\frac{\text{MeV}}{k_B T} \right)^2 = \frac{2.42\ \text{s}}{\sqrt{17.25}} \left(\frac{\text{MeV}}{80\ \text{MeV}} \right)^2 = 9.1 \times 10^{-5}\ \text{s}$$

40. It is conceivable that black holes were made in the early universe through unknown processes. Let's assume they are made after inflation, at a temperature of $k_BT = 3\times10^{15}\ \text{GeV}$**.**

(a) [6] What is the age of the universe at this time? For definiteness, use. $g_{\text{eff}} = 200$**.**

This is way into the radiation dominated era, so we have

$$t = \frac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_BT}\right)^2 = \frac{2.42\ \text{s}}{\sqrt{200}}\left(\frac{\text{MeV}}{3\times10^{18}\ \text{MeV}}\right)^2 = 1.90\times10^{-38}\ \text{s}$$

(b) [8] What is the mass density at this time, in kg/m^3?

We use the formula for mass density of radiation, namely

$$\rho = g_{\text{eff}}\,\frac{\pi^2\left(k_BT\right)^4}{30\left(\hbar c\right)^3 c^2} = \frac{200\pi^2\left(3\times10^{24}\ \text{eV}\right)^4\left(1.602\times10^{-19}\ \text{J/eV}\right)}{30\left(1.973\times10^{-7}\ \text{eV}\cdot\text{m}\right)^3\left(2.998\times10^{8}\ \text{m/s}\right)^2} = 1.24\times10^{84}\ \text{kg/m}^3$$

Due to an error on the formula sheet, you may have gotten twice this number.

(c) [6] Assume that the black holes are made from all the matter within a sphere of radius *ct*. How massive would these black holes be?

The size of the universe at this time is

$$ct = \left(1.90\times10^{-38}\ \text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right) = 5.70\times10^{-30}\ \text{m}$$

Therefore, the total mass contained in this region would be about

$$M = \tfrac{4}{3}\pi\left(ct\right)^3\rho = \tfrac{4}{3}\pi\left(5.70\times10^{-30}\ \text{m}\right)^3\left(1.24\times10^{84}\ \text{kg/m}^3\right) = 0.00096\ \text{kg} = 0.96\ \text{g}$$

It turns out such mini black holes would not survive to the current era, but it is interesting to contemplate. If you used the erroneous formula from the test, you would have gotten twice this mass.

41. Suppose some ^{10}Be were produced during primordial nucleosynthesis. This isotope has a mean lifetime of $t = 2.16\times10^{6}\ \text{yr}$.

(a) [10] Calculate the red shift z assuming the universe is matter dominated at the time. Based on the value of *z* you got, were we correct in assuming it is matter dominated?

The formula for the age for the matter dominated era is $t = (17.8\ \text{Gyr})(1+z)^{-3/2}$, so

$$(1+z)^{3/2} = \frac{17.8\ \text{Gyr}}{t} = \frac{17.8\times10^{9}\ \text{yr}}{2.16\times10^{6}\ \text{yr}} = 8241,$$

$$1+z = 8241^{\frac{2}{3}} = 408$$

Since *z* = 407, and matter/radiation equality is around *z* = 3230, we conclude that this is at a smaller value of *z* than matter-radiation equality, so we are in the matter dominated era. It works out.

(b) [10] Calculate the temperature *T* and characteristic thermal energy $k_B T$ at this time.

Temperature scales as $T = T_0(1+z)$, so

$$T = T_0(1+z) = (2.725\ \text{K})(408) = 1112\ \text{K},$$

$$k_B T = (8.671\times10^{-5}\ \text{eV/K})(1112\ \text{K}) = 0.0964\ \text{eV}.$$

42. It is conceivable that neutrinos can turn into some hypothetical particles called *Majorons* via a process $\bar{\nu}\nu \to \phi\phi$, where $\bar{\nu}$ is one type of anti-neutrino, ν is a neutrino, and ϕ is the Majoron. We want to know if this process can occur at the time when neutrinos are in thermal equilibrium with everything else, around $k_B T \approx 1\,\text{MeV}$.

(a) [6] The number density of one type of neutrino or anti-neutrino is about $n = 0.9\left(k_B T/\hbar c\right)^3$. Find the number density of neutrinos at this time, in m^{-3}.

This is straightforward:

$$n = 0.9\left(\frac{10^6\ \text{eV}}{1.973\times10^{-5}\ \text{eV}\cdot\text{m}}\right)^3 = 1.17\times10^{38}\ \text{m}^3$$

(b) [6] Estimate the age of the universe at this time ($g_{\text{eff}} = 10.75$).

This is in the radiation dominated era, so

$$t = \frac{2.42\ \text{s}}{\sqrt{\text{g}_{\text{eff}}}}\left(\frac{\text{MeV}}{k_B T}\right)^2 = \frac{2.42\ \text{s}}{\sqrt{10.75}}\left(\frac{\text{MeV}}{\text{MeV}}\right)^2 = 0.738\ \text{s}$$

(c) [8] Neutrinos were moving at speed *c* at this time. Find the minimum cross-section σ required to keep them in thermal equilibrium at this time.

To keep them in thermal equilibrium, we need to make sure every neutrino collides at least once with an anti-neutrino to make this process works. The rate of collisions is $\Gamma = n\sigma\left(\Delta v\right)$. The relative velocity for two objects moving head on would be $\Delta v = 2c$. However, it is not necessarily colliding head on, so it is not a terrible approximation to use $\Delta v = c$ instead. I'll use $\Delta v = 2c$. We want there to be one collision, so that $1 = \Gamma t$. We therefore have

$$1 = \Gamma t = n\sigma\left(\Delta v\right)t,$$

$$\sigma = \frac{1}{n\left(\Delta v\right)t} = \frac{1}{\left(1.17\times10^{38}\ \text{m}^{-3}\right)\left(2\times2.998\times10^{8}\ \text{m/s}\right)\left(0.738\ \text{s}\right)} = 1.93\times10^{-47}\ \text{m}^2$$

This is the minimum cross-section required to keep neutrinos in thermal equilibrium. It turns out that if this happens it probably messes up primordial nucleosynthesis, so this better not be taking place.

43. The current density of matter is about $\rho_{m0} = 2.54\times10^{-27}\ \text{kg/m}^3$. The local density of the disk of our galaxy is about $\rho_{\text{loc}} = 0.08\,M_\odot/\text{pc}^3$.

(a) [6] Convert the local density into kg/m³, and find the ratio $\rho_{\text{loc}}/\rho_{m0}$.

This is straightforward:

$$\rho_{\text{loc}} = \frac{\left(0.08\,M_\odot/\text{pc}^3\right)\left(1.989\times10^{30}\ \text{kg}/M_\odot\right)}{\left(3.086\times10^{16}\ \text{m/pc}\right)^3} = 5.41\times10^{-21}\ \text{kg/m}^3,$$

$$\frac{\rho_{\text{loc}}}{\rho_{m0}} = \frac{5.41\times10^{-21}\ \text{kg/m}^3}{2.54\times10^{-27}\ \text{kg/m}^3} = 2.13\times10^6.$$

(b) [8] At what red shift factor 1 + *z* did the average matter density *then* match the current density of the disk? Was this during the matter dominated era or radiation dominated era?

The density of the universe times the scale factor cubed is constant, therefore

$$\rho_m = \rho_{m0}\left(a_0/a\right)^3 = \rho_{m0}\left(1+z\right)^3,$$

$$\left(1+z\right)^3 = \rho_m/\rho_{m0} = \rho_{\text{loc}}/\rho_{m0} = 2.13\times10^6,$$

$$1+z = \left(2.13\times10^6\right)^{1/3} = 129.$$

This is well after matter/radiation equality, so it is in the matter dominated era.

(c) [6] Calculate at this time the temperature *T* and characteristic thermal energy $k_B T$.

These are simply given by

$$T = \left(1+z\right)T_0 = 129\left(2.725\ \text{K}\right) = 351\ \text{K},$$

$$k_B T = \left(8.617\times10^{-5}\ \text{eV/K}\right)\left(351\ \text{K}\right) = 0.0302\ \text{eV}.$$

44. A group of aliens living in an alternative universe discover that their universe is *very* different from ours. It has a current temperature of $T = 100\ \text{K}$, and the values of the various contributions to the total density are $\Omega_m = 0.998$, $\Omega_r = 0.001$, and $\Omega_\Lambda = 0.001$.

(a) [4] Is their universe approximately flat, or closed, or open? What currently dominates their universe?

Their total Ω is $\Omega = 0.998 + 0.001 + 0.001 = 1.000$, so it is pretty close to flat. The universe is currently matter dominated, since that is the overwhelming majority contribution to Ω.

(b) [8] Find the ratio ρ_r/ρ_m now, and in the past/future as a function of the change in scale factor a_0/a. For what value of a_0/a is this ratio approximately one? Was this in the future, or the past? What was/will be the temperature then?

The ratio is the same as the ratio of Ω's, since

$$\frac{\rho_{r0}}{\rho_{m0}}=\frac{\frac{8}{3}\pi G\rho_{r0}/H_0^2}{\frac{8}{3}\pi G\rho_{m0}/H_0^2}=\frac{\Omega_{r0}}{\Omega_{m0}}=\frac{0.001}{0.998}=\frac{1}{998}\,.$$

Because radiation scales as $\rho_r \propto a^{-4}$ and matter as $\rho_m \propto a^{-3}$, the ratio ρ_r/ρ_m scales as a^{-1}. It follows that

$$\frac{\rho_r}{\rho_m}=\frac{\rho_{r0}}{\rho_{m0}}\left(\frac{a_0}{a}\right)=\frac{a_0/a}{998}$$

It follows that matter/radiation equality occurs at $a_0/a=998$. The temperature then would be

$$T=T_0\left(a_0/a\right)=\left(100\text{ K}\right)\left(998\right)=99{,}800\text{ K}\,.$$

This would have been in the past, so the universe in the distant past would have been radiation dominated.

(c) [8] Find the ratio ρ_Λ/ρ_m now, and in the past/future as a function of the change in scale factor a_0/a. For what value of a_0/a is this ratio approximately one? Was this in the future, or the past? What was/will be the temperature then?

The ratio now is again 1/998, by the same argument. Because dark energy doesn't scale and matter scales as $\rho_m \propto a^{-3}$, the ratio ρ_Λ/ρ_m scales as a^3. It follows that

$$\frac{\rho_\Lambda}{\rho_m}=\frac{\rho_{\Lambda0}}{\rho_{m0}}\left(\frac{a}{a_0}\right)^3=\frac{\left(a_0/a\right)^{-3}}{998}$$

Dark energy would then have equality with matter when

$$a_0/a=998^{-1/3}=0.1001$$

This is in the future. Matter/vacuum energy equality will therefore be a temperature of

$$T=T_0\left(a_0/a\right)=0.1001\left(100\text{ K}\right)=10.01\text{ K}$$