Essential Tensor Equations

Changing coordinates for a tensor:

$$T^{\mu_1'\mu_2'\cdots\mu_k'}{}_{\nu_1'\nu_2'\cdots\nu_\ell'} = T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_\ell}\left(\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\right)\left(\frac{\partial x^{\mu_2'}}{\partial x^{\mu_2}}\right)\cdots\left(\frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}}\right)\left(\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\right)\left(\frac{\partial x^{\nu_2}}{\partial x^{\nu_2'}}\right)\cdots\left(\frac{\partial x^{\nu_\ell}}{\partial x^{\nu_\ell'}}\right)$$

The metric

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \quad g^{\mu\nu} = \left(g_{\mu\nu}\right)^{-1}, \quad g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu ,$$
$$g_{\mu\nu} = g_{\nu\mu}, \quad g^{\mu\nu} = g^{\nu\mu}, \quad g = \det\left(g_{\mu\nu}\right).$$

Four velocity:

$$\text{massive: } U^\mu = \frac{dx^\mu}{d\tau}, \quad g_{\mu\nu}U^\mu U^\nu = -1, \quad \text{massless: } U^\mu = \frac{dx^\mu}{d\lambda}, \quad g_{\mu\nu}U^\mu U^\nu = 0.$$

Christoffel Symbols and covariant derivatives:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right)$$
$$\nabla_\alpha T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = \partial_\alpha T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} + \Gamma^{\mu_1}_{\alpha\beta}T^{\beta\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} + \cdots + \Gamma^{\mu_k}_{\alpha\beta}T^{\mu_1\cdots\beta}{}_{\nu_1\cdots\nu_\ell}$$
$$-\Gamma^\beta_{\alpha\nu_1}T^{\mu_1\cdots\mu_k}{}_{\beta\cdots\nu_\ell} - \cdots - \Gamma^\beta_{\alpha\nu_\ell}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\beta}$$

Geodesic Equations:

$$\frac{d}{d\tau}U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0, \quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

Riemann Tensor:

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}$$
$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} = -R_{\beta\alpha\mu\nu} = R_{\mu\nu\alpha\beta}, \quad R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

Ricci Tensor and Scalar

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \partial_\alpha\Gamma^\alpha_{\nu\mu} - \partial_\nu\Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\alpha\mu}, \quad R_{\mu\nu} = R_{\nu\mu}, \quad R = g^{\mu\nu}R_{\mu\nu}$$

Einstein's equations:

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \Leftrightarrow \quad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}\right).$$