Physics 780 – General Relativity

Homework Set H

19. In homework E problem 12 we had flat 2D space $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, and then switched to polar coordinates $ds^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2$. We considered a vector $V^\mu = \left(V^x, V^y\right) = \left(A, 0\right)$, and a 1-form $V_\mu = \left(V_x, V_y\right) = \left(A, 0\right)$, where A is a constant. The Christoffel symbols in polar coordinates are $\Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = r^{-1}$, $\Gamma^\rho_{\phi\phi} = -r$, all others vanish.
    (a) Convince yourself that in the original Cartesian coordinates, all the Christoffel symbols vanish and $\nabla_\alpha V^\mu = 0$ and $\nabla_\alpha V_\mu = 0$. *This is trivial.*
    (b) Show explicitly that in polar coordinates $\nabla_\alpha V^\mu = 0$ (this is four equations).
    (c) Show explicitly that in polar coordinates $\nabla_\alpha V_\mu = 0$ (this is four equations).

20. Consider a generic 3D spherically symmetric metric, which can be written in the form

$$ds^2 = h\left(r\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2,$$

    where $h(r)$ is an unspecified function of r. It is common to abbreviate $h(r)$ as h and its derivative as h'. Our goal is to find all the non-zero components of the Christoffel symbol.
    (a) Write the metric and its inverse as a matrix (this is easy).
    (b) Argue that if $\Gamma^\nu_{\alpha\beta} \neq 0$ then an even number of indices must be ϕ.
    (c) Argue that if $\Gamma^\nu_{\alpha\beta} \neq 0$ then an even number of indices must be θ *or* there must be at least one index that is ϕ.
    (c) Calculate all non-vanishing components of $\Gamma^\nu_{\alpha\beta}$. There should be ten of them.