Physics 780 – General Relativity

Homework Set K

27. In problems 20 and 25, you had to work out a rather specific metric, but where did this metric come from? Our goal is to find the most general 3D spatial metric that is spherically symmetric; that is, one can choose two of the coordinates θ and ϕ such that the three vectors

$$L_x = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi\,, \quad L_y = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi\,, \quad L_z = \partial_\phi\,,$$

are all Killing vectors, which satisfy Killing's equation

$$K^\alpha\partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu K^\alpha + g_{\nu\alpha}\partial_\mu K^\alpha = 0.$$

We will in fact only use L_z and L_x, and will call our remaining coordinate r.

(a) Using the fact that L_z is a Killing vector, argue that all our metric components are not functions of ϕ, so $g_{\mu\nu} = g_{\mu\nu}(r,\theta)$. For parts (b) through (f), we will work with L_x.

(b) Apply Killing's equation for $\mu = \nu = r$, and show that in fact g_{rr} isn't a function of θ.

(c) Apply Killing's equation for $\mu = r, \nu = \theta$, and evaluate it at $\phi = 0$ to show that $g_{r\phi} = 0$.

(d) Apply Killing's equation for $\mu = r, \nu = \phi$ to show that $g_{r\theta} = 0$.

(e) Write Killing's equation for $\mu = \nu = \theta$, and by evaluating it at $\phi = 0$ and $\phi = \frac{1}{2}\pi$, show that $g_{\phi\theta} = 0$ and $g_{\theta\theta}$ is not a function of θ.

(f) Apply Killing's equation for $\mu = \theta, \nu = \phi$ to show that $g_{\phi\phi} = \sin^2\theta\, g_{\theta\theta}$.

(g) At this point, the metric must take the form $ds^2 = a(r)\,\mathrm{d}r^2 + b(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$. Change variables $r \to r'$, where $r' = \sqrt{b(r)}$. What is the form of the metric now? If you need it, just let b^{-1} be the inverse function of b.