Physics 780 – General Relativity

Homework Set N

34. Consider a light beam approaching a black hole with mass M. The light beam is moving in the plane $\theta = \frac{1}{2}\pi$. From class notes, we have

$$\frac{E^2}{J^2} - \frac{1}{r^2} + \frac{2GM}{r^3} = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 .$$

(a) Find the radius at which the left side of this equation is extremized (minimum or maximum). Is it a minimum or maximum? A photon at this radius can circle endlessly, ($dr/d\phi = 0$) if E/J has the right value. Will this be a stable or unstable orbit?

(b) What will be the value of E/J for this orbit? Keeping in mind that we showed in class that $J/E = b$ is the impact parameter, you should be able to find the impact parameter b_C which will end up converging to a circular orbit.

(c) Qualitatively, what will happen to a photon that starts at a larger impact parameter, so we have $b > b_C$? That is to say, will there be any radius where $dr/d\phi = 0$? What if $b < b_C$?

(d) Find the cross-section for absorption of photons by a black hole; that is, the area of incoming photons that are absorbed by the black hole.

35. Can you make a black hole in two spatial dimensions? We will work in polar coordinates (t, r, ϕ), and assume the stress-energy tensor is zero away from the origin.

(a) First, what is the flat spacetime metric in polar coordinates? If you don't know, write it in Cartesian coordinates and rewrite it using $x = r\cos\phi$ and $y = r\sin\phi$.

(b) Assume the black hole is rotationally invariant and time invariant, so we can choose coordinates such that ∂_t and ∂_ϕ are Killing vectors. What does this tell us about the metric components, *i.e.*, what coordinates can they depend on?

(c) Assume the metric is invariant under reflection so that $\phi \to -\phi$. Argue that the metric now takes the form $ds^2 = -f\,\mathrm{d}t^2 + 2j\,\mathrm{d}r\,\mathrm{d}t + h\,\mathrm{d}r^2 + b\,\mathrm{d}\phi^2$.

(d) Change time to a new coordinate $t \to t' = t - \int (j/f)\,dr$. Show that this eliminates j. Once you have done so, rename any functions and variables so the metric now takes the form in part (c), but with $j = 0$.

(e) Explain why you can change variables r such that the metric is now $ds^2 = -f\,\mathrm{d}t^2 + h\,\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2$.

(f) Find all the components of the Ricci tensor and/or Einstein tensor. I recommend you use **grcalc** or a similar method to save your sanity.

(g) Since we have no source away from the origin, $R_{\mu\nu} = G_{\mu\nu} = 0$. Based on this, show that f and h must both be constants (I used the Einstein tensor). Argue that by rescaling your time coordinate, one of these functions can be set equal to one.

(h) Show that by rescaling the radial and angular coordinate $r' = r\sqrt{h}$ and $\phi' = \phi/\sqrt{h}$, we can make the metric look just like the one in part (a). You might think this metric is identical to flat spacetime, but it is not. Why not? Hint: what is the range of ϕ'?