Physics 780 – General Relativity

Homework Set P

40. The Tolman-Oppenheimer-Volkoff equations, $\frac{dp}{dr} = -\frac{G\left(M(r) + 4\pi pr^3\right)(\rho + p)}{r\left(r - 2GM(r)\right)}$, are generally hard to solve analytically, but we will do so for an idealized situation, which is called an infinitely stiff equation of state, where $\rho(r) = \begin{cases} \rho_0 & \text{if } r < R, \\ 0 & \text{if } r > R, \end{cases}$ and ρ_0 is a constant.

Comment: When writing equations by hand, I tend to write p as P so it doesn't look like ρ.

(a) Find a formula for $M(r)$, the integrated mass, for $r < R$. What is the total mass $M = M(R)$?

(b) Rearrange the TOV equation so that the left side has only functions of p and the right side has only functions of r. It will look like $f(p)\,dp = g(r)\,dr$.

(c) Integrate equation (b) to get a relationship between p and r. Don't forget the constant of integration! The constant of integration will be chosen here or in part (d) so that the pressure at $r = R$ is zero.

(d) Do some work to solve the result of eq. (c) for the pressure p as a function of r. The terms with G in them can be simplified by eliminating ρ_0 in favor of the total mass M and the total radius R.

(e) The pressure should be highest at the center. Write the pressure at this point. Find the largest radius R for fixed M such that the pressure is finite at the origin, $p(0) < \infty$.