Physics 780 – General Relativity

Homework Set U

49. Way back in the previous millennium (i.e., pre-1995) we only knew about matter (and a tiny bit of radiation), and were none too confident about the value of Ω_m. For *this* problem, assume the universe contains matter only.
   (a) Show that if $\Omega_m \le 1$, the universe will never stop growing, *i.e.*, there is no time in the future when $\dot{a} = 0$.
   (b) Show that if $\Omega_m > 1$, it is inevitable that the universe will eventually stop growing. Find a formula for the size of the universe compared to now, a/a_0, when the universe will stop growing as a function of Ω_m.

50. Suppose in some gauge choice, an almost-flat universe has metric perturbations that satisfy the harmonic condition, $\partial_\mu h^\mu{}_\beta(x) = \frac{1}{2}\partial_\beta h^\mu{}_\mu(x)$.
   (a) Show that if we make a small coordinate change $x^\mu \to x'^\mu + \xi^\mu$, the harmonic condition is still preserved if $\Box\xi_\mu = 0$.
   (b) Suppose we are looking at wave solutions $h_{\mu\nu}(x) = h_{\mu\nu} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$, with $|\mathbf{k}| = \omega$. Show that the coordinate change $\xi_0 = (ih_{00}/2\omega) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$ will cause $h'_{00} = 0$.
   (c) Show that a subsequent coordinate change $\xi_i = (ih_{i0}/\omega) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$ will cause $h'_{i0} = 0$.

51. We defined gravity waves by writing $h_{\mu\nu}(x) = h_{\mu\nu} \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$, and then writing h in terms of two polarization vectors $h_{\mu\nu} = h_+ e^+_{\mu\nu} + h_\times e^\times_{\mu\nu}$. These are not the *only* choices for the basis tensors for gravity waves. Let's assume **k** is in the *z*-direction.
   (a) Define the right-helicity and left-helicity vectors as $e^R_{\mu\nu} = e^+_{\mu\nu} + ie^\times_{\mu\nu}$ and $e^L_{\mu\nu} = e^+_{\mu\nu} - ie^\times_{\mu\nu}$. Show that any wave can be written in the form $h_{\mu\nu} = h_R e^R_{\mu\nu} + h_L e^L_{\mu\nu}$, and find formulas for $h_{R,L}$ in terms of $h_{+,\times}$ and vice-versa.
   (b) Consider a wave containing only the right-helicity wave (*i.e.* $h_L = 0$). Perform a rotation of this wave around the *z*-axis by an angle θ, with inverse Lorentz transformation

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   Show that such a transformation simply multiplies the metric perturbation $h_{\mu\nu}$ by a phase $e^{im\theta}$ and determine the value of the *helicity m*. Note that Lorentz transforms on indices that are down work as $h'_{\mu\nu} = h_{\alpha\beta}\left(\Lambda^{-1}\right)^\alpha{}_\mu \left(\Lambda^{-1}\right)^\beta{}_\nu$. Repeat for the left-helicity wave.