Physics 780 – General Relativity

Homework Set V

52. In class we demonstrated that a general gravity wave can be written as $h_{\mu\nu}(x) = h_{\mu\nu} e^{ik\cdot x} + h^*_{\mu\nu} e^{-ik\cdot x}$, where $h_{\mu\nu}$ is a complex tensor which is assumed to be spacelike only $h_{0\nu} = 0$, and transverse so $k_\mu h^{\mu\nu} = 0$. Let's assume it is traveling in the +*z* direction, so $k^\mu = (k, 0, 0, k)$.
    (a) Substitute this expression into the formula for the gravitational stress-energy in this case, $8\pi G t_{\mu\nu} = -\frac{1}{2} h^{\alpha\sigma} \partial_\mu \partial_\nu h_{\alpha\sigma} - \frac{1}{4} \partial_\mu h^{\alpha\sigma} \partial_\nu h_{\alpha\sigma}$.
    (b) Some of the terms now have no space dependance, and others go like $e^{\pm 2ik\cdot x}$. Argue that the terms like $e^{\pm 2ik\cdot x}$ will average to zero if you time average (what is the value of cos(2*ωt*) and sin(2*ωt*).
    (c) Write an explicit expression for the time-averaged value of $\langle t^{30} \rangle$ in terms of $h_{\mu\nu}$ and $h^*_{\mu\nu}$.
    (d) If we write $h_{\mu\nu} = h^+ e^+_{\mu\nu} + h^\times e^\times_{\mu\nu}$, write $\langle t^{30} \rangle$ explicitly in terms of h^+ and $h^\times$.

53. In class we found the following expressions for the magnitude of the gravitational waves in terms of quadrupole moments:

$$h^{00} = k_i k_j Q^{ij} + \omega^2 Q^{ii}, \quad h^{0i} = h^{i0} = 2Q^{ij}\omega k_j, \quad h^{ij} = 2\omega^2 Q^{ij} + \delta^{ij}\left(k_\ell k_m Q^{\ell m} - \omega^2 Q^{\ell\ell}\right)$$

    The four-vector *k* is given by $k^\mu = (\omega, \mathbf{k})$, with $\omega = |\mathbf{k}|$.
    (a) As a warm-up, find the trace $h^\mu{}_\mu = \eta_{\mu\nu} h^{\mu\nu}$.
    (b) We now want to start checking the harmonic condition $k_\mu h^{\mu\nu} = \frac{1}{2} k^\nu h^\mu{}_\mu$. Check this for the time component $\nu = 0$.
    (c) Now check it for the space components, $\nu = j$.