Physics 780 – General Relativity

Solution Set B

5. [15] We can define the four-acceleration as $A^{\mu} = \frac{d}{d\tau}U^{\mu} = \frac{d^2}{d\tau^2}x^{\mu}$**.**

(a) [4] Show that if you are in a frame such that a particle is momentarily at rest, so $U^{\mu}(\tau_0) = (1,0,0,0)$**, then** $\eta_{\mu\nu}A^{\mu}A^{\nu} = \mathbf{a}^2$**, where a is the ordinary acceleration** $\mathbf{a} = \frac{d\mathbf{v}}{dt}$**. Since** $a = \sqrt{\eta_{\mu\nu}A^{\mu}A^{\nu}}$ **is Lorentz invariant, this must be the magnitude of the acceleration as experienced by an object with arbitrary velocity** U^{μ}**.**

The four-velocity is given by $U^{\mu}(\tau_0) = (\gamma, \gamma\mathbf{v})$, where $\gamma = (1-v^2)^{-1/2}$. Taking the derivative, we have

$$A^{\mu} = \frac{d}{d\tau}U^{\mu} = \left(\frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau}\mathbf{v} + \gamma\frac{d\mathbf{v}}{d\tau}\right) = \frac{dt}{d\tau}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\mathbf{v} + \gamma\frac{d\mathbf{v}}{dt}\right).$$

It is easy to show that $d\gamma/dt = (1-v^2)^{-3/2} v(dv/dt) = \gamma^3 v(dv/dt)$. We also know that $dt/d\tau = \gamma$. So we have

$$A^{\mu} = \left(\gamma^4 v\frac{dv}{dt}, \gamma^4 \mathbf{v} v\frac{dv}{dt} + \gamma^2\mathbf{a}\right).$$

If the instantaneous velocity is zero, then we have $v = 0$, $\gamma = 1$, and therefore $A^{\mu} = (0, \mathbf{a})$, so $\eta_{\mu\nu}A^{\mu}A^{\nu} = \mathbf{a}^2$. Hence $\sqrt{\eta_{\mu\nu}A^{\mu}A^{\nu}} = a$.

(b) [5] An object moves according to the formula $x = \sqrt{b^2+t^2} - b$**,** $y = z = 0$**. Rewrite these in terms of proper time** τ **(hint: this was almost done for you in class), then work out the four-velocity** $U^{\mu}(\tau)$ **the four-acceleration** $A^{\mu}(\tau)$ **and find a formula for the proper acceleration *a*, which should be constant.**

This is almost identical with the problem done in class. We have $dx = t\,dt\big/\sqrt{b^2+t^2}$, so

$$\tau = \int d\tau = \int\sqrt{-\eta_{\mu\nu}dx^{\mu}dx^{\nu}} = \int\sqrt{dt^2 - dx^2} = \int\sqrt{dt^2 - \frac{(t\,dt)^2}{b^2+t^2}} = \int\frac{b\,dt}{\sqrt{b^2+t^2}}.$$

This integral is identical to one we did in class, and is given by

$$\tau = b\sinh^{-1}(t/b).$$

We can invert this to get *t* in terms of *τ*, and then substitute to find *x* in terms of *τ*:

$$t = b\sinh(\tau/b),$$
$$x = \sqrt{b^2+t^2} - b = \sqrt{b^2 + b^2\sinh^2(\tau/b)} - b = b\cosh(\tau/b) - b.$$

We now start working out the four-velocity and the four-acceleration:

$$U^\mu = \frac{d}{d\tau}\left(b\sinh(\tau/b), b\cosh(\tau/b)-b, 0, 0\right) = \left(\cosh(\tau/b), \sinh(\tau/b), 0, 0\right)$$
$$A^\mu = \frac{d}{d\tau}\left(\cosh(\tau/b), \sinh(\tau/b), 0, 0\right) = \left(\frac{1}{b}\sinh\left(\frac{\tau}{b}\right), \frac{1}{b}\cosh\left(\frac{\tau}{b}\right), 0, 0\right).$$

The acceleration dotted into itself is therefore

$$\eta_{\mu\nu}A^\mu A^\nu = -\left(A^0\right)^2 + \left(A^1\right)^2 = -\frac{1}{b^2}\sinh^2\left(\frac{\tau}{b}\right) + \frac{1}{b^2}\cosh^2\left(\frac{\tau}{b}\right) = \frac{1}{b^2}.$$

We therefore conclude that $a^2 = 1/b^2$, so $b = 1/a$.

(c) [3] Determine the value of *b* in years (or light-years) if $a = g = 9.80\ \text{m/s}^2$.

Keeping in mind that we are working in units where $c = 1$, we have

$$b = \frac{1}{g} = \frac{2.998\times10^8\ \text{m/s}}{9.80\ \text{m/s}^2} = \frac{3.059\times10^7\ \text{s}}{3.156\times10^7\ \text{s/yr}} = 0.969\ \text{yr}.$$

Since a light year is c·yr, this is also the same in light years.

(d) [3] If you leave Earth, starting at rest, and accelerate at *g*, how much proper time in years would it take you go get to α-Centauri (4.3 *c*·yr), the center of the galaxy (27,000 *c*·yr) and the approximate edge of the Universe (2×10^{10} *c*·yr).

We note that our formulas have initially $x = 0$, so these formulas describe the motion as a function of proper time. Solving for τ in terms of x, we find

$$\tau = b\cosh^{-1}\left(\frac{x}{b}+1\right).$$

We now simply substitute all the relevant distances

$$\tau_\alpha = (0.969\ \text{yr})\cosh^{-1}\left(\frac{4.30\ \text{yr}}{0.969\ \text{yr}}+1\right) = 2.30\ \text{yr}, \quad \tau_g = (0.969\ \text{yr})\cosh^{-1}\left(\frac{27{,}000\ \text{yr}}{0.969\ \text{yr}}+1\right) = 10.6\ \text{yr},$$
$$\tau_U = (0.969\ \text{yr})\cosh^{-1}\left(\frac{2\times10^{10}\ \text{yr}}{0.969\ \text{yr}}+1\right) = 23.7\ \text{yr}.$$

Interestingly, you can get almost anywhere in the universe, but you have to use a substantial fraction of your lifetime to get there.

6. The electromagnetic field tensor $F_{\mu\nu}$ is given by equation (1.69). If the fields are given by the three components of E and B, what would be the new values of the electric and magnetic fields E′ and B′ if you

(a) Performed a rotation by an angle θ around the z-axis,

(b) Perform a boost in the x-direction by rapidity ϕ.

The corresponding inverse Lorentz matrices in each case are given below. Note that because $F_{\mu\nu}$ is anti-symmetric, you can calculate just six components of $F_{\mu\nu}$ to get E′ and B′.

$$\Lambda(\theta)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda(\phi)^{-1} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We simply start working them all out, using the inverse Lorenz transformations. For the rotations, we have

$$E_1' = F_{10}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{1}\left(\Lambda^{-1}\right)^{\nu}{}_{0} = F_{10}\left(\Lambda^{-1}\right)^{1}{}_{1}\left(\Lambda^{-1}\right)^{0}{}_{0} + F_{20}\left(\Lambda^{-1}\right)^{2}{}_{1}\left(\Lambda^{-1}\right)^{0}{}_{0} = E_1\cos\theta + E_2\sin\theta,$$

$$E_2' = F_{20}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{2}\left(\Lambda^{-1}\right)^{\nu}{}_{0} = F_{10}\left(\Lambda^{-1}\right)^{1}{}_{2}\left(\Lambda^{-1}\right)^{0}{}_{0} + F_{20}\left(\Lambda^{-1}\right)^{2}{}_{2}\left(\Lambda^{-1}\right)^{0}{}_{0} = -E_1\sin\theta + E_2\cos\theta,$$

$$E_3' = F_{30}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{3}\left(\Lambda^{-1}\right)^{\nu}{}_{0} = F_{30}\left(\Lambda^{-1}\right)^{3}{}_{3}\left(\Lambda^{-1}\right)^{0}{}_{0} = E_3,$$

$$B_1' = F_{23}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{2}\left(\Lambda^{-1}\right)^{\nu}{}_{3} = F_{13}\left(\Lambda^{-1}\right)^{1}{}_{2}\left(\Lambda^{-1}\right)^{3}{}_{3} + F_{23}\left(\Lambda^{-1}\right)^{2}{}_{2}\left(\Lambda^{-1}\right)^{3}{}_{3} = B_2\sin\theta + B_1\cos\theta,$$

$$B_2' = F_{31}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{3}\left(\Lambda^{-1}\right)^{\nu}{}_{1} = F_{31}\left(\Lambda^{-1}\right)^{3}{}_{3}\left(\Lambda^{-1}\right)^{1}{}_{1} + F_{32}\left(\Lambda^{-1}\right)^{3}{}_{3}\left(\Lambda^{-1}\right)^{2}{}_{1} = B_2\cos\theta - B_1\sin\theta,$$

$$B_3' = F_{12}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{1}\left(\Lambda^{-1}\right)^{\nu}{}_{2} = F_{12}\left(\Lambda^{-1}\right)^{1}{}_{1}\left(\Lambda^{-1}\right)^{2}{}_{2} + F_{21}\left(\Lambda^{-1}\right)^{2}{}_{1}\left(\Lambda^{-1}\right)^{1}{}_{2} = B_3\cos^2\theta + B_3\sin^2\theta$$
$$= B_3 .$$

For the boosts we have

$$E_1' = F_{10}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{1}\left(\Lambda^{-1}\right)^{\nu}{}_{0} = F_{10}\left(\Lambda^{-1}\right)^{1}{}_{1}\left(\Lambda^{-1}\right)^{0}{}_{0} + F_{01}\left(\Lambda^{-1}\right)^{0}{}_{1}\left(\Lambda^{-1}\right)^{1}{}_{0} = E_1\left(\cosh^2\phi - \sinh^2\phi\right)$$
$$= E_1,$$

$$E_2' = F_{20}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{2}\left(\Lambda^{-1}\right)^{\nu}{}_{0} = F_{20}\left(\Lambda^{-1}\right)^{2}{}_{2}\left(\Lambda^{-1}\right)^{0}{}_{0} + F_{21}\left(\Lambda^{-1}\right)^{2}{}_{2}\left(\Lambda^{-1}\right)^{1}{}_{0} = E_2\cosh\phi - B_3\sinh\phi,$$

$$E_3' = F_{30}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{3}\left(\Lambda^{-1}\right)^{\nu}{}_{0} = F_{30}\left(\Lambda^{-1}\right)^{3}{}_{3}\left(\Lambda^{-1}\right)^{0}{}_{0} + F_{31}\left(\Lambda^{-1}\right)^{3}{}_{3}\left(\Lambda^{-1}\right)^{1}{}_{0} = E_3\cosh\phi + B_2\sinh\phi,$$

$$B_1' = F_{23}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{2}\left(\Lambda^{-1}\right)^{\nu}{}_{3} = F_{23}\left(\Lambda^{-1}\right)^{2}{}_{2}\left(\Lambda^{-1}\right)^{3}{}_{3} = B_1,$$

$$B_2' = F_{31}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{3}\left(\Lambda^{-1}\right)^{\nu}{}_{1} = F_{30}\left(\Lambda^{-1}\right)^{3}{}_{3}\left(\Lambda^{-1}\right)^{0}{}_{1} + F_{31}\left(\Lambda^{-1}\right)^{3}{}_{3}\left(\Lambda^{-1}\right)^{1}{}_{1} = E_3\sinh\phi + B_2\cosh\phi,$$

$$B_3' = F_{12}' = F_{\mu\nu}\left(\Lambda^{-1}\right)^{\mu}{}_{1}\left(\Lambda^{-1}\right)^{\nu}{}_{2} = F_{02}\left(\Lambda^{-1}\right)^{0}{}_{1}\left(\Lambda^{-1}\right)^{2}{}_{2} + F_{12}\left(\Lambda^{-1}\right)^{1}{}_{1}\left(\Lambda^{-1}\right)^{2}{}_{2} = -E_2\sinh\phi + B_3\cosh\phi.$$