Physics 780 – General Relativity

Solution Set F

14. The metric in flat 3D space is $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. Show that in spherical coordinates, this is given by $ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$. Spherical coordinates are defined by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

We simply dive in and start calculating.

$$\begin{aligned}
ds^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \left[\mathrm{d}\left(r\sin\theta\cos\phi\right)\right]^2 + \left[\mathrm{d}\left(r\sin\theta\sin\phi\right)\right]^2 + \left[\mathrm{d}\left(r\cos\theta\right)\right]^2 \\
&= \left(\sin\theta\cos\phi\,\mathrm{d}r + r\cos\theta\cos\phi\,\mathrm{d}\theta - r\sin\theta\sin\phi\,\mathrm{d}\phi\right)^2 \\
&\quad + \left(\sin\theta\sin\phi\,\mathrm{d}r + r\cos\theta\sin\phi\,\mathrm{d}\theta + r\sin\theta\sin\phi\,\mathrm{d}\phi\right)^2 + \left(\cos\theta\ \mathrm{d}r - r\sin\theta\ \mathrm{d}\theta\right)^2 \\
&= \left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right)\mathrm{d}r^2 + r^2\left(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right)\mathrm{d}\theta^2 \\
&\quad + r^2\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right)\mathrm{d}\phi^2 + 2r\sin\theta\cos\theta\left(\sin^2\phi + \cos^2\phi - 1\right)\mathrm{d}r\,\mathrm{d}\theta \\
&\quad + 2r\sin^2\theta\cos\phi\sin\phi\left(-1+1\right)\mathrm{d}r\,\mathrm{d}\phi + 2r^2\sin\theta\cos\theta\cos\phi\sin\phi\left(-1+1\right)\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \left(\sin^2\theta + \cos^2\theta\right)\mathrm{d}r^2 + r^2\left(\cos^2\theta + \sin^2\theta\right)\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\ \mathrm{d}\phi^2.
\end{aligned}$$

That was painful, but it worked out in the end.

15. Consider the 3D metric $ds^2 = \left(1-r^2\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$. This has an *apparent* singularity at $r = 0$, because the determinant $g = 0$ there, but we understand that that's just a coordinate singularity at a point. What about the apparent singularity at $r = 1$, where $g = \infty$?

(a) By looking, for example, at the circle defined by $r \rightarrow 1$, $\theta = \frac{1}{2}\pi$, $\phi = \left(0, 2\pi\right)$ argue that those points with $r = 1$ are decidedly not just a point.

We are moving in a circle with fixed r and fixed θ, so we have $ds^2 = r^2\sin^2\theta\,\mathrm{d}\phi^2 \to \mathrm{d}\phi^2$. Since $\phi = \left(0, 2\pi\right)$, this is a circle of radius 1 with circumference 2π, and hence clearly not a point.

(b) Make the substitution $r = \sin\psi$, while keeping the coordinates θ and ϕ. Show that the resulting metric no longer has a singularity at $r = 1$ (now at $\psi = \frac{1}{2}\pi$).

We know that $\mathrm{d}r = \mathrm{d}\left(\sin\psi\right) = \cos\psi\,\mathrm{d}\psi$. Substituting in, we have

$$ds^2 = \frac{\left(\sin\psi\,\mathrm{d}\psi\right)^2}{1-\sin^2\psi} + \sin^2\psi\,\mathrm{d}\theta^2 + \sin^2\psi\sin^2\theta\,\mathrm{d}\phi^2 = \mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\theta^2 + \sin^2\psi\sin^2\theta\,\mathrm{d}\phi^2.$$

The value at $r = 1$ corresponds to $\psi = \frac{1}{2}\pi$, but the metric is perfectly well-behaved at this point, having a finite and non-vanishing determinant. Hence the original coordinate system just has a breakdown of coordinates at $r = 1$, it isn't a real singularity.

(c) There is now an apparent singularity where the metric has problems at $\psi = \pi$. Convince yourself that this is, in fact, just a point.

At $\psi = \pi$, the metric has zero determinant, which means, for example, that it no longer has an inverse. But if you look at the set of points $\psi = \pi$, it is clear that the coefficients of $\mathrm{d}\theta$ and $\mathrm{d}\phi$ are now zero. This suggests that this is really just a point (much as $\psi = 0$ is just a point), so it is just a failure of coordinates at this point.

16. Given a metric, is it curved? The answer isn't always obvious. Consider the metric

$$ds^2 = -\mathrm{d}t^2 + t^2\left(\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right)$$

We will convert to coordinates $(t,r,\theta,\phi) \to (T,R,\theta,\phi)$, defined by

$$\begin{Bmatrix} T = t\sqrt{r^2+1} \\ R = rt \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} t = \sqrt{T^2 - R^2} \\ r = \dfrac{R}{\sqrt{T^2-R^2}} \end{Bmatrix}$$

with θ and ϕ the same in both coordinate systems

(a) Write $\mathrm{d}t$ and $\mathrm{d}r$ in terms of $\mathrm{d}T$ and $\mathrm{d}R$.

We simply start computing:

$$\mathrm{d}t = \frac{T\,\mathrm{d}T - R\,\mathrm{d}R}{\sqrt{T^2-R^2}},$$

$$\mathrm{d}r = \frac{\mathrm{d}R}{\sqrt{T^2-R^2}} - \frac{RT\mathrm{d}T - R^2\,\mathrm{d}R}{\left(T^2-R^2\right)^{3/2}} = \frac{\left(T^2-R^2\right)\mathrm{d}R - RT\mathrm{d}T + R^2\,\mathrm{d}R}{\left(T^2-R^2\right)^{3/2}} = \frac{T^2\mathrm{d}R - RT\mathrm{d}T}{\left(T^2-R^2\right)^{3/2}}.$$

(b) Write the metric out entirely in the new coordinates. If you make no mistake, there should be no cross-terms and the coefficients should all be simple.

To save a little work, we notice that the combination rt is just R. For the rest, we simply blindly substitute to yield

$$
\begin{aligned}
ds^2 &= -\frac{\left(T\mathrm{d}T - R\mathrm{d}R\right)^2}{T^2 - R^2} + \frac{T^2 - R^2}{1 + \frac{R^2}{T^2 - R^2}} \frac{\left(T^2\mathrm{d}R - RT\mathrm{d}T\right)^2}{\left(T^2 - R^2\right)^3} + R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 \\
&= -\frac{\left(T\mathrm{d}T - R\mathrm{d}R\right)^2}{T^2 - R^2} + \frac{\left(T^2\mathrm{d}R - RT\mathrm{d}T\right)^2}{T^2\left(T^2 - R^2\right)} + R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 \\
&= \frac{\left(T\mathrm{d}R - R\mathrm{d}T\right)^2 - \left(T\mathrm{d}T - R\mathrm{d}R\right)^2}{T^2 - R^2} + R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 \\
&= \frac{\left(T^2 - R^2\right)\mathrm{d}R^2 + \left(R^2 - T^2\right)\mathrm{d}T^2}{T^2 - R^2} + R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 \\
&= -\mathrm{d}T^2 + \mathrm{d}R^2 + R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 .
\end{aligned}
$$

(c) By comparison with the metric in problem 14, argue this is simply disguised flat spacetime.

The space part is just flat space in spherical coordinate. The first term is just the time for flat spacetime.