Physics 780 – General Relativity

Solution Set G

17. Consider the flat FLRW metric, $ds^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$.

(a) Consider first the case of a radiation-dominated universe, $a(t) = \sqrt{t}$, with a big bang singularity at *t* = 0. In time *t*, how far can a light beam travel, starting at the origin? Give your answer in the form $s = kt$, where *k* is a simple constant.

Light beams travel at $d\tau^2 = -ds^2 = 0$, so if it moves in the *x*-direction, this implies $dx/dt = 1/a$. Integrating, we have

$$x = \int \frac{dt}{a(t)} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t}.$$

The physical distance is $s = \int a\,dx = ax = 2\sqrt{t}\sqrt{t} = 2t$.

(b) Now consider an exponentially expanding universe, with $a(t) = e^{Ht}$, with *H* a constant. In this case, nothing special happens at *t* = 0, so let's define *t* = 0 as now. Imagine a light beam starting at us at *x* = 0 and traveling in the *x*-direction. Find *x*(*t*), and show that there is a limiting value x_∞ that cannot be reached by the light beam, even as $t \to \infty$.

We simply do the same integral again, but with the new function, which yields

$$x = \int \frac{dt}{a(t)} = \int e^{-Ht}dt = H^{-1}\left(1 - e^{-Ht}\right),$$

where the constant of integration was chosen to assure that $x(t=0) = 0$. This has a limiting value of $x = H^{-1}$. Note that since $a(0) = 1$, this is the physical distance now to the object. In summary, anything that is at a distance of $x = H^{-1}$ can never be affected by us, and similarly, something at this location now can never affect us.

18. In this problem we will find the 2D "volume" of two similar metrics. Note that the answer is not guaranteed to be finite.

(a) First consider the metric $ds^2 = \dfrac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left(1 + x^2 + y^2\right)^2}$, where *x* and *y* are unrestricted real numbers. As a first step, rewrite this metric in polar coordinates, $(x, y) = (\rho\cos\phi, \rho\sin\phi)$. What is the appropriate range of ρ and ϕ?

We first note that

$$\begin{aligned}
\mathrm{d}x^2+\mathrm{d}y^2 &= \left(\mathrm{d}\left(\rho\cos\phi\right)\right)^2+\left(\mathrm{d}\left(\rho\sin\phi\right)\right)^2 \\
&= \left(\cos\phi\,\mathrm{d}\rho-\rho\sin\phi\;\mathrm{d}\phi\right)^2+\left(\sin\phi\,\mathrm{d}\rho+\rho\cos\phi\,\mathrm{d}\phi\right)^2 \\
&= \left(\cos^2\phi+\sin^2\phi\right)\mathrm{d}\rho^2+\left(\rho^2\sin^2\phi+\rho^2\cos^2\phi\right)\mathrm{d}\phi^2+2\rho\sin\phi\cos\phi\left(1-1\right)\mathrm{d}\rho\,\mathrm{d}\phi \\
&= \mathrm{d}\rho^2+\rho^2\mathrm{d}\phi^2
\end{aligned}$$

Substituting this into the metric, we have

$$ds^2=\frac{\mathrm{d}\rho^2+\rho^2\mathrm{d}\phi^2}{\left(1+\rho^2\right)^2}.$$

It is clear that the range for ρ is $\left(0,\infty\right)$, and to get in all directions, the range for ϕ is $\left(0,2\pi\right)$.

(b) Calculate the volume of the metric described in part (a).

We first find the determinant of the metric and take the square root. The determinant is

$$g=g_{\rho\rho}g_{\phi\phi}=\frac{1}{\left(1+\rho^2\right)^2}\cdot\frac{\rho^2}{\left(1+\rho^2\right)^2},$$

$$\sqrt{|g|}=\frac{\rho}{\left(1+\rho^2\right)^2}$$

We now integrate this over the relevant coordinates, so we have

$$V=\int\sqrt{|g|}d^2x=\int_0^\infty\frac{\rho\,d\rho}{\left(1+\rho^2\right)^2}\int_0^{2\pi}d\phi=2\pi\cdot\left[-\tfrac{1}{2}\left(1+\rho^2\right)^{-1}\right]_0^\infty=2\pi\cdot\tfrac{1}{2}=\pi.$$

(c) Repeat parts (a) and (b) for the metric $ds^2=\dfrac{\mathrm{d}x^2+\mathrm{d}y^2}{\left(1-x^2-y^2\right)^2}$**, where now *x* and *y* are restricted to the disk** $x^2+y^2<1$**.**

The work for the metric is essentially identical, so we have

$$ds^2=\frac{\mathrm{d}\rho^2+\rho^2\mathrm{d}\phi^2}{\left(1-\rho^2\right)^2},\quad\sqrt{g}=\frac{\rho}{\left(1-\rho^2\right)^2}.$$

Although the range for ϕ is still $\left(0,2\pi\right)$, the range for ρ is now $\left(0,1\right)$. The volume is now

$$V=\int\sqrt{|g|}d^2x=\int_0^1\frac{\rho\,d\rho}{\left(1-\rho^2\right)^2}\int_0^{2\pi}d\phi=2\pi\left[\tfrac{1}{2}\left(1-\rho^2\right)^{-1}\right]_0^1=\pi\left(\infty-1\right)=\infty.$$