Physics 780 – General Relativity

Solution Set H

19. In homework E problem 12 we had flat 2D space $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, and then switched to polar coordinates $ds^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2$. We considered a vector $V^\mu = \left(V^x, V^y\right) = \left(A, 0\right)$, and a 1-form $V_\mu = \left(V_x, V_y\right) = \left(A, 0\right)$, where *A* is a constant. The Christoffel symbols in polar coordinates are $\Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = \rho^{-1}$, $\Gamma^\rho_{\phi\phi} = -\rho$, all others vanish.

(a) Convince yourself that in the original Cartesian coordinates, all the Christoffel symbols vanish and $\nabla_\alpha V^\mu = 0$ and $\nabla_\alpha V_\mu = 0$. *This is trivial.*

In Cartesian coordinates, the metric has vanishing derivatives, so $\Gamma^\alpha_{\mu\nu} = 0$. Therefore $\nabla_\alpha V^\mu = \partial_\alpha V^\mu = 0$ and $\nabla_\alpha V_\mu = \partial_\alpha V_\mu = 0$.

(b) Show explicitly that in polar coordinates $\nabla_\alpha V^\mu = 0$ (this is four equations).

The vector and covector appear in the solutions to problem 12. We simply calculate all four components using $\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma^\mu_{\alpha\beta} V^\beta$:

$$\nabla_\rho V^\rho = \partial_\rho V^\rho + 0 = \partial_\rho \left(A\cos\phi\right) = 0,$$

$$\nabla_\rho V^\phi = \partial_\rho V^\phi + \Gamma^\phi_{\rho\phi} V^\phi = \partial_\rho \left(-\frac{A\sin\phi}{\rho}\right) + \frac{1}{\rho}\left(-\frac{A\sin\phi}{\rho}\right) = \frac{A\sin\phi}{\rho^2} - \frac{A\sin\phi}{\rho^2} = 0,$$

$$\nabla_\phi V^\rho = \partial_\phi V^\rho + \Gamma^\rho_{\phi\phi} V^\phi = \partial_\phi \left(A\cos\phi\right) - \rho\left(-\frac{A\sin\phi}{\rho}\right) = -A\sin\phi + A\sin\phi = 0,$$

$$\nabla_\phi V^\phi = \partial_\phi V^\phi + \Gamma^\phi_{\phi\rho} V^\rho = \partial_\phi \left(-\frac{A\sin\phi}{\rho}\right) + \frac{1}{\rho} A\cos\phi = -\frac{A\cos\phi}{\rho} + \frac{A\cos\phi}{\rho} = 0.$$

(c) Show explicitly that in polar coordinates $\nabla_\alpha V_\mu = 0$ (this is four equations).

We simply use the formula $\nabla_\alpha V_\mu = \partial_\alpha V_\mu - \Gamma^\mu_{\alpha\beta} V_\mu$ to show

$$\nabla_\rho V_\rho = \partial_\rho V_\rho - 0 = \partial_\rho \left(A\cos\phi\right) = 0,$$

$$\nabla_\rho V_\phi = \partial_\rho V_\phi - \Gamma^\phi_{\rho\phi} V_\phi = \partial_\rho \left(-A\rho\sin\phi\right) - \frac{1}{\rho}\left(-A\rho\sin\phi\right) = -A\sin\phi + A\sin\phi = 0,$$

$$\nabla_\phi V_\rho = \partial_\phi V_\rho - \Gamma^\phi_{\phi\rho} V_\phi = \partial_\phi \left(A\cos\phi\right) - \frac{1}{\rho}\left(-A\rho\sin\phi\right) = -A\sin\phi + A\sin\phi = 0,$$

$$\nabla_\phi V_\phi = \partial_\phi V_\phi - \Gamma^\rho_{\phi\phi} V^\rho = \partial_\phi \left(-A\rho\sin\phi\right) + \rho\left(A\cos\phi\right) = -A\rho\cos\phi + A\rho\cos\phi = 0.$$

That was boring, but it came out to zero as expected.

20. [15] Consider a generic 3D spherically symmetric metric, which can be written in the form

$$ds^2 = h(r)\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\,,$$

where $h(r)$ is an unspecified function of r. It is common to abbreviate $h(r)$ as h and its derivative as h'. Our goal is to find all the non-zero components of the Christoffel symbol.

(a) [2] Write the metric and its inverse as a matrix (this is easy).

The metric and inverse metric are

$$g_{\mu\nu} = \mathrm{diag}\left(h, r^2, r^2\sin^2\theta\right) \quad \text{and} \quad g^{\mu\nu} = \mathrm{diag}\left(\frac{1}{h}, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right).$$

(b) [3] Argue that if $\Gamma^\nu_{\alpha\beta} \neq 0$ then an even number of indices must be ϕ.

The connection is given by $\Gamma^\nu_{\alpha\beta} = \frac{1}{2}g^{\nu\mu}\left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)$. Noting that the metric and its inverse are invertible, all indices must occur in pairs, *except* for the derivative index. But nothing in the metric depends on ϕ, so any term with ∂_ϕ will automatically vanish. Hence ϕ is only on the metric factors, which come in pairs, so $\Gamma^\nu_{\alpha\beta}$ will have an even number (0 or 2) ϕ's.

(c) [3] Argue that if $\Gamma^\nu_{\alpha\beta} \neq 0$ then an even number of indices must be θ *or* there must be at least one index that is ϕ.

The argument is almost identical, except that one component, $g_{\phi\phi}$ does depend on θ. Hence the only way to have an odd number of θ's, it must also have at least one ϕ. Combining this with part (b), the conclusion is that the only connections with an odd number of θ's will have two ϕ's and one θ.

(d) [7] Calculate all non-vanishing components of $\Gamma^\nu_{\alpha\beta}$. There should be ten of them.

We simply start work on all the remaining possibilities, saving some time by using the symmetry of the lower two indices.

$$\Gamma^r_{rr} = \tfrac{1}{2}g^{rr}\left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) = \frac{h'}{2h},$$

$$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \tfrac{1}{2}g^{\theta\theta}\left(\partial_r g_{\theta\theta}\right) = \frac{1}{2r^2}\partial_r\left(r^2\right) = \frac{1}{r},$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \tfrac{1}{2}g^{\phi\phi}\left(\partial_r g_{\phi\phi}\right) = \frac{1}{2r^2\sin^2\theta}\partial_r\left(r^2\sin^2\theta\right) = \frac{1}{r},$$

$$\Gamma^r_{\theta\theta} = \tfrac{1}{2} g^{rr}\left(-\partial_r g_{\theta\theta}\right) = -\frac{1}{2h}\partial_r\left(r^2\right) = -\frac{r}{h},$$

$$\Gamma^r_{\phi\phi} = \tfrac{1}{2} g^{rr}\left(-\partial_r g_{\phi\phi}\right) = -\frac{1}{2h}\partial_r\left(r^2\sin^2\theta\right) = -\frac{r\sin^2\theta}{h},$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \tfrac{1}{2} g^{\phi\phi}\left(\partial_\theta g_{\phi\phi}\right) = \frac{1}{2r^2\sin^2\theta}\partial_\theta\left(r^2\sin^2\theta\right) = \cot\theta,$$

$$\Gamma^\theta_{\phi\phi} = \tfrac{1}{2} g^{\theta\theta}\left(-\partial_\theta g_{\phi\phi}\right) = -\frac{1}{2r^2}\partial_\theta\left(r^2\sin^2\theta\right) = -\sin\theta\cos\theta.$$

Since we found ten, this is probably correct. To summarize, the results are

$$\Gamma^r_{rr} = \frac{h'}{2h}, \quad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^r_{\theta\theta} = -\frac{r}{h}, \quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{h},$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta.$$