Physics 780 – General Relativity

Solution Set J

25. In homework set H, problem 20, you had to work out all the components if $\Gamma^{\nu}_{\alpha\beta}$ for the metric $ds^2 = h(r)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$.

(a) Use these to get all non-zero components of the Riemann tensor of the form $R^{\mu}{}_{\nu\mu\nu}$ (no sums). There should be six in total. As a check, note that they must all vanish if $h(r)=1$.

The indices μ and ν take on three values each, so you might think that there would be nine possible components of this form. However, the anti-symmetry of the last two indices guarantees that it will automatically vanish if $\mu=\nu$. We now just start working out the six remaining cases, using the formula $R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu\Gamma^{\alpha}_{\nu\beta} - \partial_\nu\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta}$.

$$R^{r}{}_{\theta r\theta} = \partial_r\Gamma^{r}_{\theta\theta} - \partial_\theta\Gamma^{r}_{r\theta} + \Gamma^{r}_{r\alpha}\Gamma^{\alpha}_{\theta\theta} - \Gamma^{r}_{\theta\alpha}\Gamma^{\alpha}_{r\theta} = -\partial_r\left(\frac{r}{h}\right) - \frac{h'}{2h}\left(\frac{r}{h}\right) + \left(\frac{r}{h}\right)\frac{1}{r} = \frac{h'r}{h^2} - \frac{1}{h} - \frac{h'r}{2h^2} + \frac{1}{h} = \frac{h'r}{2h^2},$$

$$R^{r}{}_{\phi r\phi} = \partial_r\Gamma^{r}_{\phi\phi} - \partial_\phi\Gamma^{r}_{r\phi} + \Gamma^{r}_{r\alpha}\Gamma^{\alpha}_{\phi\phi} - \Gamma^{r}_{\phi\alpha}\Gamma^{\alpha}_{r\phi} = -\partial_r\left(\frac{r\sin^2\theta}{h}\right) - \frac{h'}{2h}\left(\frac{r\sin^2\theta}{h}\right) + \left(\frac{r\sin^2\theta}{h}\right)\frac{1}{r}$$

$$= \frac{h'r\sin^2\theta}{h^2} - \frac{\sin^2\theta}{h} - \frac{h'r\sin^2\theta}{2h^2} + \frac{\sin^2\theta}{h} = \frac{h'r\sin^2\theta}{2h^2},$$

$$R^{\theta}{}_{r\theta r} = \partial_\theta\Gamma^{\theta}_{rr} - \partial_r\Gamma^{\theta}_{\theta r} + \Gamma^{\theta}_{\theta\alpha}\Gamma^{\alpha}_{rr} - \Gamma^{\theta}_{r\alpha}\Gamma^{\alpha}_{\theta r} = -\partial_r\left(\frac{1}{r}\right) + \frac{1}{r}\frac{h'}{2h} - \frac{1}{r}\cdot\frac{1}{r} = \frac{h'}{2rh},$$

$$R^{\phi}{}_{r\phi r} = \partial_\phi\Gamma^{\phi}_{rr} - \partial_r\Gamma^{\phi}_{\phi r} + \Gamma^{\phi}_{\phi\alpha}\Gamma^{\alpha}_{rr} - \Gamma^{\phi}_{r\alpha}\Gamma^{\alpha}_{\phi r} = -\partial_r\left(\frac{1}{r}\right) + \frac{1}{r}\frac{h'}{2h} - \frac{1}{r}\cdot\frac{1}{r} = \frac{h'}{2rh},$$

$$R^{\theta}{}_{\phi\theta\phi} = \partial_\theta\Gamma^{\theta}_{\phi\phi} - \partial_\phi\Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{\theta\alpha}\Gamma^{\alpha}_{\phi\phi} - \Gamma^{\theta}_{\phi\alpha}\Gamma^{\alpha}_{\theta\phi} = -\partial_\theta\left(\sin\theta\cos\theta\right) - \frac{1}{r}\left(\frac{r\sin^2\theta}{h}\right) + \sin\theta\cos\theta\cot\theta$$

$$= -\cos^2\theta + \sin^2\theta - \frac{\sin^2\theta}{h} + \cos^2\theta = \sin^2\theta\left(1 - \frac{1}{h}\right),$$

$$R^{\phi}{}_{\theta\phi\theta} = \partial_\phi\Gamma^{\phi}_{\theta\theta} - \partial_\theta\Gamma^{\phi}_{\theta\phi} + \Gamma^{\phi}_{\phi\alpha}\Gamma^{\alpha}_{\theta\theta} - \Gamma^{\phi}_{\theta\alpha}\Gamma^{\alpha}_{\phi\theta} = -\partial_\theta\cot\theta - \frac{1}{r}\left(\frac{r}{h}\right) - \cot^2\theta = \csc^2\theta - \frac{1}{h} - \cot^2\theta = 1 - \frac{1}{h}.$$

(b) Find the diagonal components of the Ricci tensor, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, for the three components R_{rr}, $R_{\theta\theta}$, and $R_{\phi\phi}$. If you have made no mistakes so far, you should find $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$.

This works out pretty easily, as we just have two terms to add in each case

$$R_{rr} = R^{\alpha}{}_{r\alpha r} = R^{r}{}_{rrr} + R^{\theta}{}_{r\theta r} + R^{\phi}{}_{r\phi r} = \frac{h'}{2rh} + \frac{h'}{2rh} = \frac{h'}{rh},$$
$$R_{\theta\theta} = R^{\alpha}{}_{\theta\alpha\theta} = R^{r}{}_{\theta r\theta} + R^{\theta}{}_{\theta\theta\theta} + R^{\phi}{}_{\theta\phi\theta} = \frac{h'r}{2h^2} + 1 - \frac{1}{h},$$
$$R_{\phi\phi} = R^{\alpha}{}_{\phi\alpha\phi} = R^{r}{}_{\phi r\theta} + R^{\theta}{}_{\phi\theta\phi} + R^{\phi}{}_{\phi\phi\phi} = \frac{h'r\sin^2\theta}{2h^2} + \left(1 - \frac{1}{h}\right)\sin^2\theta.$$

Obviously, $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$.

(c) Find the Ricci scalar and show that it equals $R = \frac{2h'}{rh^2} + \frac{2}{r^2} - \frac{2}{r^2 h}$.

We have

$$R = g^{\mu\nu}R_{\mu\nu} = g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{1}{h}\frac{h'}{rh} + \frac{1}{r^2}\left(\frac{h'r}{2h^2} + 1 - \frac{1}{h}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{h'r}{2h^2} + 1 - \frac{1}{h}\right)\sin^2\theta$$
$$= \frac{h'}{rh^2} + \frac{h'}{h^2 r} + \frac{2}{r^2} - \frac{2}{r^2 h} = \frac{2h'}{h^2 r} + \frac{2}{r^2} - \frac{2}{r^2 h}.$$

26. Assume that the metric found in question 24 is homogenous, and in particular, the Ricci scalar is a constant given by 6*C*, so $R = 6C$.

(a) Find a simple formula for the combination $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r}{h}\right)$.

First, setting $R = 6C$, we have $\frac{h'}{h^2 r} + \frac{1}{r^2} - \frac{1}{r^2 h} = 3C$. Expanding the combination, we have

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r}{h}\right) = \frac{1}{r^2}\left(\frac{1}{h} - \frac{rh'}{h^2}\right) = \frac{1}{r^2 h} - \frac{h'}{rh^2} = \frac{1}{r^2} - 3C.$$

(b) Multiply this equation by r^2 and integrate it. The constant of integration can be found if we insist that $h(r)$ does not vanish at the origin. Solve the equation for h.

Multiplying by r^2 and integrating, we have

$$\frac{d}{dr}\left(\frac{r}{h}\right) = 1 - 3Cr^2, \quad \text{so} \quad \frac{r}{h} = \int\left(1 - 3Cr^2\right)dr = r - Cr^3 + k.$$

Assuming h is non-zero at the origin, the left side vanishes, and the right side vanishes only if k = 0, so we pick $k = 0$, then solve for h:

$$h = \frac{r}{r - Cr^3} = \frac{1}{1 - Cr^2}.$$