Physics 780 – General Relativity

Homework Set K

27. In problems 20 and 25, you had to work out a rather specific metric, but where did this metric come from? Our goal is to find the most general 3D spatial metric that is spherically symmetric; that is, one can choose two of the coordinates θ and ϕ such that the three vectors

$$L_x = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi, \quad L_y = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi, \quad L_z = \partial_\phi,$$

are all Killing vectors, which satisfy Killing's equation

$$K^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu K^\alpha + g_{\nu\alpha}\partial_\mu K^\alpha = 0.$$

We will in fact only use *L_z* and *L_x*, and will call our remaining coordinate *r*.

(a) Using the fact that L_z is a Killing vector, argue that all our metric components are not functions of ϕ, so $g_{\mu\nu} = g_{\mu\nu}(r,\theta)$.

If you write out Killing's equation for the K_z, it turns into $\partial_\phi g_{\mu\nu} = 0$, so $g_{\mu\nu} = g_{\mu\nu}(r,\theta)$.

(b) Apply Killing's equation for $\mu = \nu = r$, and show that in fact g_{rr} isn't a function of θ.

The vector L_x has no r dependance, so the derivative terms acting on L_x will vanish. Keeping in mind that $\partial_\phi g_{\mu\nu} = 0$, we see that $L_x^\alpha \partial_\alpha g_{\mu\nu} = -\sin\phi\partial_\theta g_{\mu\nu}$ in general. Setting $\mu = \nu = r$, Klling's equation is now $-\sin\phi\partial_\theta g_{rr} = 0$, so g_{rr} isn't a function of θ, $g_{rr} = g_{rr}(r)$.

(c) Apply Killing's equation for $\mu = r, \nu = \theta$, and evaluate it at $\phi = 0$ to show that $g_{r\phi} = 0$.

We write out the equation as

$$\begin{aligned} 0 &= -\sin\phi\partial_\theta g_{r\theta} + g_{r\alpha}\partial_\theta L_x^\alpha + g_{\theta\alpha}\partial_r L_x^\alpha = -\sin\phi\partial_\theta g_{r\theta} - g_{r\phi}\partial_\theta(\cos\phi\cot\theta) \\ &= -\sin\phi\,\partial_\theta g_{r\theta} + g_{r\phi}\cos\phi\csc^2\theta. \end{aligned}$$

Evaluating at $\phi = 0$, we have $g_{r\phi}\csc^2\theta = 0$, so $g_{r\phi} = 0$.

(d) Apply Killing's equation for $\mu = r, \nu = \phi$ to show that $g_{r\theta} = 0$.

We do similar work to show

$$0 = -\sin\phi\partial_\theta g_{r\phi} + g_{r\alpha}\partial_\phi L_x^\alpha + g_{\phi\alpha}\partial_r L_x^\alpha = 0 + g_{r\theta}\partial_\phi(-\sin\phi) = g_{r\theta}\cos\phi.$$

We see that $g_{r\theta}=0$.

(e) Write Killing's equation for $\mu=\nu=\theta$, and by evaluating it at $\phi=0$ and $\phi=\frac{1}{2}\pi$, show that $g_{\theta\phi}=0$ and $g_{\theta\theta}$ is not a function of θ.

We have

$$0=\cos\phi\partial_\theta g_{\theta\theta}+2g_{\theta\alpha}\partial_\theta L_y^\alpha=-\sin\phi\partial_\theta g_{\theta\theta}-2g_{\theta\phi}\partial_\theta\left(\cos\phi\cot\theta\right)=-\sin\phi\partial_\theta g_{\theta\theta}+2g_{\theta\phi}\cos\phi\csc^2\theta.$$

Setting $\phi=0$, we have $2g_{\theta\phi}\csc^2\theta=0$, or $g_{\theta\phi}=0$. Setting $\phi=\frac{1}{2}\pi$, we have $\partial_\theta g_{\theta\theta}=0$.

(f) Apply Killing's equation for $\mu=\theta,\nu=\phi$ to show that $g_{\phi\phi}=\sin^2\theta\, g_{\theta\theta}$.

Here we have

$$\begin{aligned}0&=\cos\phi\partial_\theta g_{\theta\phi}+g_{\theta\alpha}\partial_\phi L_y^\alpha+g_{\phi\alpha}\partial_\theta L_y^\alpha=g_{\theta\theta}\partial_\phi\left(\cos\phi\right)+g_{\phi\phi}\partial_\theta\left(-\sin\phi\cot\theta\right)\\&=-g_{\theta\theta}\sin\phi+g_{\phi\phi}\sin\phi\csc^2\theta,\\g_{\phi\phi}&=g_{\theta\theta}\sin^2\theta.\end{aligned}$$

(g) At this point, the metric must take the form $ds^2=a(r)\mathrm{d}r^2+b(r)\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)$. Change variables $r\to r'$, where $r'=\sqrt{b(r)}$. What is the form of the metric now? If you need it, just let b^{-1} be the inverse function of *b*.

Of course, when you do this, the *b*(*r*) term becomes just r'^2. Solving the equation for *r*, we have $r=b^{-1}\left(r'^2\right)$. We then have

$$\mathrm{d}r=\mathrm{d}b^{-1}\left(r'^2\right)=\frac{d}{dr'}b^{-1}\left(r'^2\right)\mathrm{d}r'=2r'b^{-1\prime}\left(r'^2\right)\mathrm{d}r'.$$

Substituting this into the given metric, we would have

$$ds^2=4r'^2a\left(b^{-1}\left(r'^2\right)\right)\left(b^{-1\prime}\left(r'^2\right)\right)^2\mathrm{d}r'^2+r'^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right).$$

Since we have no idea what the functions *a* or *b* are, we can just call the horrendous first term *h*(*r*′), and then we can rename *r*′ → *r* to rewrite this in the standard form

$$ds^2=h(r)\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right).$$