Physics 780 – General Relativity

Solution Set N

34. [10] Consider a light beam approaching a black hole with mass *M*. The light beam is moving in the plane $\theta = \frac{1}{2}\pi$. From class notes, we have

$$\frac{E^2}{J^2} - \frac{1}{r^2} + \frac{2GM}{r^3} = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 .$$

(a) [5] Find the radius at which the left side of this equation is extremized (minimum or maximum). Is it a minimum or maximum? A photon at this radius can circle endlessly, ($dr/d\phi = 0$) if *E*/*J* has the right value. Will this be a stable or unstable orbit?

We simply take the derivative of the left side and set it to zero, so

$$0 = \frac{d}{dr}\left(\frac{E^2}{J^2} - \frac{1}{r^2} + \frac{2GM}{r^3}\right) = \frac{2}{r^3} - \frac{6GM}{r^4} = \frac{2}{r^4}\left(r - 3GM\right).$$

This will occur at $R = 3GM$. By taking another derivative and evaluating it at this value, we find

$$\frac{d^2}{dr^2}\left(\frac{E^2}{J^2} - \frac{1}{r^2} + \frac{2GM}{r^3}\right)\Bigg|_{3GM} = \left(-\frac{6}{r^4} + \frac{24GM}{r^5}\right)\Bigg|_{3GM} = -\frac{6}{81G^4M^4} + \frac{24}{243G^4M^4} = \frac{2}{81G^4M^4} > 0.$$

Because the second derivative is positive, it is a local maximum, and hence this is an unstable orbit.

(b) [5] What will be the value of *E*/*J* for this orbit? Keeping in mind that we showed in class that *J*/*E* = *b* is the impact parameter, you should be able to find the impact parameter b_C which will end up converging to a circular orbit.

Since it is in a circular orbit, we have $dr/d\phi = 0$. Therefore we have

$$0 = \frac{E^2}{J^2} - \frac{1}{r^2} + \frac{2GM}{r^3} = \frac{E^2}{J^2} - \frac{1}{(3GM)^2} + \frac{2GM}{(3GM)^3} = \frac{E^2}{J^2} - \frac{1}{9G^2M^2} + \frac{2}{27G^3M^3} = \frac{E^2}{J^2} - \frac{1}{27G^2M^2},$$

$$\frac{E}{J} = \frac{1}{\sqrt{27}GM}.$$

Flipping this upside down, we see that we have $b_C = \sqrt{27}GM$.

(c) [3] Qualitatively, what will happen to a photon that starts at a larger impact parameter, so we have $b > b_C$? That is to say, will there be any radius where $dr/d\phi = 0$? What if $b < b_C$?

If $b > b_C$, then E/J will be smaller, and it will be impossible to reach this critical radius $R = 3GM$. Hence a photon will instead stop at some minimum radius and then leave again, and it will miss the black hole. If $b < b_C$, then the photon will get sucked into the black hole.

(d) [2] Find the cross-section for absorption of photons by a black hole; that is, the area of incoming photons that are absorbed by the black hole.

The cross-section is just the total area of the region that absorbs photons, which is $\pi b_C^2 = 27\pi G^2 M^2$. The naïve cross suction would just be the area of a circle the size of the event horizon, which is $\pi R_S^2 = 4\pi G^2 M^2$, so the actual cross-section is 6.75 times larger than this.

35. [25] Can you make a black hole in two spatial dimensions? We will work in polar coordinates (t, r, ϕ), and assume the stress-energy tensor is zero away from the origin.

(a) [2] First, what is the flat spacetime metric in polar coordinates? If you don't know, write it in Cartesian coordinates and rewrite it using $x = r\cos\phi$ and $y = r\sin\phi$.

We have done this several times before in two dimensions, and adding the third time dimension doesn't change much, so we can show $ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2$.

(b) [2] Assume the black hole is rotationally invariant and time invariant, so we can choose coordinates such that ∂_t and ∂_ϕ are Killing vectors. What does this tell us about the metric components, *i.e.*, what coordinates can they depend on?

All of the metric components will be independent of both t and ϕ, so they will depend only on r.

(c) [3] Assume the metric is invariant under reflection so that $\phi \to -\phi$. Argue that the metric now takes the form $ds^2 = -f\,\mathrm{d}t^2 + 2j\,\mathrm{d}r\;\mathrm{d}t + h\,\mathrm{d}r^2 + b\;\mathrm{d}\phi^2$.

In general, all nine components (six independent) of the metric could exist. However, terms like $k\,\mathrm{d}r\,\mathrm{d}\phi$ or $m\,\mathrm{d}t\,\mathrm{d}\phi$ would go to their negatives under the reflection symmetry proposed, and hence must vanish. These are the only terms that could remain.

(d) [4] Change time to a new coordinate $t \to t' = t - \int (j/f)\,dr$. Show that this eliminates *j*. Once you have done so, rename any functions and variables so the metric now takes the form in part (c), but with *j* = 0.

We define t' as indicated, and then find that $dt = dt' + (j/f)\,dr$. Substituting in, the metric is now

$$ds^2 = -f\left(\mathrm{d}t' + (j/f)\mathrm{d}r\right)^2 + 2j\,\mathrm{d}r\left(\mathrm{d}t' + (j/f)\mathrm{d}r\right) + h\,\mathrm{d}r^2 + b\,\mathrm{d}\phi^2$$
$$= -f\mathrm{d}t'^2 + \left(h + j^2/f\right)\mathrm{d}r^2 + b\,\mathrm{d}\phi^2 .$$

We define $h' = h + j^2/f$, and suddenly the metric takes the same form as before, but with a couple of primes thrown in. We then rename $h' \to h$ and $t' \to t$, and we have it.

(e) [3] Explain why you can change variables *r* such that the metric is now $ds^2 = -f\,\mathrm{d}t^2 + h\,\mathrm{d}r^2 + r^2\mathrm{d}\phi^2$.

We simply define a new radial coordinate $r' = \sqrt{b(r)}$, and the effect is that the $\mathrm{d}\phi^2$ term now will just be multiplied by r'^2. This will change the functions *f* and *h*, of course, but since we don't know what they are, we just rename the new functions as *f* and *h* and rename *r′* as *r*.

(f) [3] Find all the components of the Ricci tensor and/or Einstein tensor. I recommend you use grcalc or a similar method to save your sanity.

I used **grcalc** and decided to focus on the Einstein tensor, since this was simplest. I found

$$G_{tt} = \frac{fh'}{2h^2}, \quad G_{rr} = \frac{f'}{2fr}, \quad G_{\phi\phi} = \frac{r^2 f''}{2fh} - \frac{rf'^2}{4f^2h} - \frac{r^2 f'h'}{4fh^2}.$$

(g) [4] Since we have no source away from the origin, $R_{\mu\nu} = G_{\mu\nu} = 0$. Based on this, show that *f* and *h* must both be constants (I used the Einstein tensor). Argue that by rescaling your time coordinate, one of these functions can be set equal to one.

The equation $G_{tt} = 0$ tells us that $h' = 0$ so that *h* is constant. The equation $G_{rr} = 0$ tells us that $f' = 0$ so that *f* is constant. We can then define a new time coordinate $t' = t\sqrt{f}$, and then rewrite the metric in terms of *t′*, which makes the metric $ds^2 = -\mathrm{d}t'^2 + h\,\mathrm{d}r^2 + r^2\mathrm{d}\phi^2$.

(h) [4] Show that by rescaling the radial and angular coordinate $r' = r\sqrt{h}$ and $\phi' = \phi/\sqrt{h}$, we can make the metric look just like the one in part (a). You might think this metric is identical to empty spacetime, but it is not. Why not? Hint: what is the range of ϕ'?

It is obvious that $\mathrm{d}r'^2 = h\mathrm{d}r^2$. We also find $r'^2\mathrm{d}\phi'^2 = hr^2\,\mathrm{d}\phi^2/h = r^2\mathrm{d}\phi^2$. We therefore have a metric $ds^2 = -\mathrm{d}t'^2 + \mathrm{d}r'^2 + r'^2\mathrm{d}\phi'^2$, which, other than the primes, is identical with the metric in part (a). However, even though the range of *t′* and *r′* are still their usual ranges of $t' \in (-\infty, \infty)$ and $r' \in (0, \infty)$, as usual, the range for $\phi' \in \left(0, 2\pi/\sqrt{h}\right)$. Assuming $h > 1$, it is a flat universe with a wedge removed.