Physics 780 – General Relativity

Solution Set O

36. In homework set L, problem 30, you found the general solution for a black hole if there is also a cosmological constant. In the problem, we are going to consider a universe with no black hole and just a cosmological constant, with metric

$$ds^2 = -\left(1-\tfrac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 + \left(1-\tfrac{1}{3}\Lambda r^2\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

Our ultimate goal is to change coordinates to get rid of the apparent singularity, and make a Penrose diagram for this metric.

(a) This metric has an apparent singularity at *r* = *b* (what is *b*?). Rewrite the metric in terms of *b* instead of Λ. In which regions of radius $r \in (0,\infty)$ are *r* and *t* spacelike or timelike?

We start by noting that we have apparent singularities at $b = \sqrt{3/\Lambda}$, and we can write

$$ds^2 = -\left(1 - r^2/b^2\right)\mathrm{d}t^2 + \left(1 - r^2/b^2\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

It is obvious from this formulation that *r* is spacelike and *t* timelike for $r < b$ but the two switch for $r > b$. At the moment, we only trust the metric for $r < b$, but will ultimately extend it so it works everywhere.

(b) As we did for Schwarzschild, define a coordinate $r^* = r^*(r)$ such that light-like radial curves will have $dr^*/dt = \pm 1$, *i.e.*, at 45° angles. This will require an integration; choose the constant of integration so that $r^* = 0$ when *r* = 0. What value of r^* corresponds to the trouble spot *r* = *b*?

A light beam will have $ds = 0$, and if it is moving radially then $d\theta = d\phi = 0$. It is easy to show that such a beam will therefore satisfy $dr/dt = \pm\left(1 - r^2/b^2\right)$. We want to define a tortoise coordinate so that they will satisfy $dr^*/dt = \pm 1$, which suggests $dr^*/dr = \left(1 - r^2/b^2\right)^{-1}$. To find r^*, we simply integrate this, so

$$r^* = \int \frac{dr^*}{dr}dr = \int \frac{dr}{1 - r^2/b^2} = b^2 \frac{1}{b}\tanh^{-1}\left(\frac{r}{b}\right) = b\tanh^{-1}\left(\frac{r}{b}\right)$$

It is easy to invert this equation, so that we also have $r = b\tanh\left(r^*/b\right)$. The constant of integration was chosen as suggested in the problem.

(c) Unlike Schwarzschild, it is easy to invert this relation, so we can find $r = r\left(r^*\right)$. Use this to write the metric entirely in terms of r^*. Then change variables to null coordinates $t, r^* \to u, v$, where $v = t + r^*$ and $u = t - r^*$. In *u*, *v* coordinates, where is *r* = 0 now? In *u*, *v* coordinates, where is *r* = *b* now? Write the metric in terms of *u* and *v*.

The change to r^* was made such that the $\mathrm{d}r^{*2}$ term will have the same coefficient (except for sign) as the $\mathrm{d}t^2$ term. We therefore have

$$\begin{aligned} ds^2 &= \left[1-\tanh^2\left(r^*/b\right)\right]\left(\mathrm{d}r^{*2}-\mathrm{d}t^2\right)+b^2\tanh^2\left(r^*/b\right)\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right) \\ &= \operatorname{sech}^2\left(r^*/b\right)\left(\mathrm{d}r^{*2}-\mathrm{d}t^2\right)+b^2\tanh^2\left(r^*/b\right)\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right). \end{aligned}$$

We then make the change to *v* and *u* as suggested, noting that $\mathrm{d}u\,\mathrm{d}v = \mathrm{d}t^2 - \mathrm{d}r^{*2}$. We also note that $r^* = \tfrac{1}{2}(v-u)$, so we have

$$ds^2 = -\operatorname{sech}^2\left(\frac{v-u}{2b}\right)\mathrm{d}u\,\mathrm{d}v + b^2\tanh^2\left(\frac{v-u}{2b}\right)\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right).$$

We note first that since $\tanh^{-1}(1) = \infty$, $r = b$ corresponds to $r^* = \infty$, and therefore $v = \infty$ and $u = -\infty$. In contrast, $r = 0$ is $r^* = 0$, and this is the line $v = u$.

(d) In an attempt to get *r* = *b* back under control, define new coordinates $v' = -e^{-v/b}$ and $u' = e^{u/b}$. Write the metric in terms of *u′* and *v′*. Write a formula for *r* in terms of *u′* and *v′*. Write the metric in terms of *u′* and *v′*. What is the equation for the points that correspond to *r* = 0? To *r* = *b*? To *r* = ∞?

I found it easiest to invert these two coordinate changes, so $v = -b\ln(-v')$ and $u = b\ln(u')$. We therefore would have $du/\mathrm{d}u' = b/u'$ and $dv/dv' = -b/v'$ so that $\mathrm{d}u\,\mathrm{d}v = -b^2\mathrm{d}u'\mathrm{d}v'/u'v'$. Notice that, for example, $e^{\pm u/2b} = u'^{\pm 1/2}$ and similarly $e^{\pm v/2b} = (-v')^{\mp 1/2}$. Substituting in, we find

$$\begin{aligned} ds^2 &= -\frac{4}{\left(e^{v/2b}e^{-u/2b}+e^{-v/2b}e^{u/2b}\right)^2}\frac{b^2\mathrm{d}u'\,\mathrm{d}v'}{(-u'v')}+b^2\left(\frac{e^{v/2b}e^{-u/2b}-e^{-v/2b}e^{u/2b}}{e^{v/2b}e^{-u/2b}+e^{-v/2b}e^{u/2b}}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right) \\ &= -\frac{4b^2\mathrm{d}u'\,\mathrm{d}v'}{(-u'v')\left(1/\sqrt{-u'v'}+\sqrt{-u'v'}\right)^2}+b^2\left(\frac{1/\sqrt{-u'v'}-\sqrt{-u'v'}}{1/\sqrt{-u'v'}+\sqrt{-u'v'}}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right) \\ &= -\frac{4b^2\mathrm{d}u'\,\mathrm{d}v'}{(1-u'v')^2}+b^2\left(\frac{1+u'v'}{1-u'v'}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right). \end{aligned}$$

It was a mess for a while, but it simplified rather nicely. Now, we want to know what the various coordinates correspond to. We know that the coefficient of $\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)$ is r^2, so obviously

$$r=b\frac{1+u'v'}{1-u'v'}$$

The points corresponding to $r = 0$ are when $1+u'v'=0$, which is the double hyperbola $u'v'=-1$. The points corresponding to $r = b$ are when $u'v'=0$, which is the crossed lines $u'=0$ and $v'=0$. And infinity comes from when the denominator vanishes, or $u'v'=+1$.

(e) Define new coordinates $u'=\tan u''$, $v'=\tan v''$. Write the metric in terms of u'' and v''. For this final step, eliminate *b* and go back to Λ for the metric.

For this step, we note that the overall metric is just proportional to $b^2=3/\Lambda$, so we just put that on the outside. These formulas are trivial to invert, $u'=\tan u''$ and $v'=\tan v''$, so we find

$$\begin{aligned}
ds^2&=\frac{3}{\Lambda}\left[-\frac{4\sec^2 u''\sec^2 v''\,\mathrm{d}u''\,\mathrm{d}v''}{\left(1-\tan u''\tan v''\right)^2}+\left(\frac{1+\tan u''\tan v''}{1-\tan u''\tan v''}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)\right]\\
&=\frac{3}{\Lambda}\left[-\frac{4\mathrm{d}u''\,\mathrm{d}v''}{\left(\cos u''\cos v''-\sin u''\sin v''\right)^2}+\left(\frac{\cos u''\cos v''+\sin u''\sin v''}{\cos u''\cos v''-\sin u''\sin v''}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)\right]\\
&=\frac{3}{\Lambda}\frac{-4\mathrm{d}u''\,\mathrm{d}v''+\cos^2\left(u''-v''\right)\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)}{\cos^2\left(u''+v''\right)}
\end{aligned}$$

Again, it was a mess at intermediate steps, but it wasn't so bad in the end. The final formula for r is

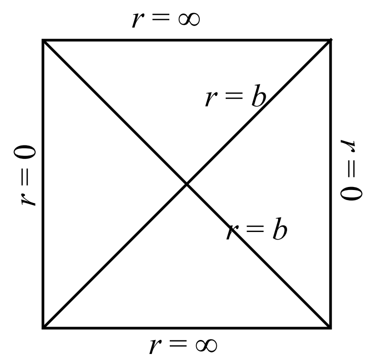
$$r=b\frac{1+\tan\left(u''\right)\tan\left(v''\right)}{1-\tan\left(u''\right)\tan\left(v''\right)}=b\frac{\cos\left(u''\right)\cos\left(v''\right)+\sin\left(u''\right)\sin\left(v''\right)}{\cos\left(u''\right)\cos\left(v''\right)-\sin\left(u''\right)\sin\left(v''\right)}=b\frac{\cos\left(u''-v''\right)}{\cos\left(u''+v''\right)}.$$

(f) Make a final change of coordinates to $u'',v''\to R,T$, where $v''=\frac{1}{2}\left(T+R\right)$ and $u''=\frac{1}{2}\left(T-R\right)$. Write the metric in terms of *T* and *R*. In (*T*,*R*) space, where are the locations *r* = 0, *b*, and ∞? Make a Penrose diagram in (*T*,*R*) coordinates, with these three values of *r* marked as one or more lines.

It is pretty easy so see that $\mathrm{d}v''\,\mathrm{d}u''=\frac{1}{4}\left(\mathrm{d}R^2-\mathrm{d}T^2\right)$, that $u''-v''=-R$ and $u''+v''=T$, so

$$ds^2=\frac{3}{\Lambda}\cdot\frac{-dT^2+dR^2+\cos^2\left(R\right)\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)}{\cos^2\left(T\right)}.$$

The radial coordinate is given by $\frac{r}{b} = \frac{\cos(R)}{\cos(T)}$. From this we can see that $r = 0$ happens when $\cos(R) = 0$, which is $R = \pm\frac{1}{2}\pi$, $r = \infty$ is when $\cos(T) = 0$, which is when $T = \pm\frac{1}{2}\pi$, and $r = b$ when $\cos(R) = \cos(T)$, which, since cosine is an even function, happens when $R = \pm T$. At right is the Penrose diagram, a square with the left and right boundaries corresponding to $r = 0$, top and bottom to $r = \infty$, and the two diagonals are $r = b$.



Possibly Helpful Formulas: $\int \frac{dx}{b^2 - x^2} = \frac{1}{b}\tanh^{-1}\left(\frac{x}{b}\right)$, $\frac{d}{d\psi}\tanh\psi = \operatorname{sech}^2\psi$

$$\tanh^2\psi + \operatorname{sech}^2\psi = 1, \quad \tanh\psi = \frac{e^{\psi} - e^{-\psi}}{e^{\psi} + e^{-\psi}}, \quad \operatorname{sech}\psi = \frac{2}{e^{\psi} + e^{-\psi}},$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$