Physics 780 – General Relativity

Solution Set P

40. The Tolman-Oppenheimer-Volkoff equations, $\frac{dp}{dr} = -\frac{G\left(M(r)+4\pi pr^3\right)(\rho+p)}{r\left(r-2GM(r)\right)}$, are generally hard to solve analytically, but we will do so for an idealized situation, which is called an infinitely stiff equation of state, where $\rho(r) = \begin{cases} \rho_0 & \text{if } r < R, \\ 0 & \text{if } r > R, \end{cases}$ and ρ_0 is a constant. Comment: When writing equations by hand, I tend to write *p* as *P* so it doesn't look like ρ.

(a) Find a formula for $M(r)$, the integrated mass, for *r* < *R*. What is the total mass $M = M(R)$?

The equation for the mass is trivial to integrate,

$$M(r) = 4\pi\int \rho(r) r^2 dr = 4\pi\rho_0 \int r^2 dr = \tfrac{4}{3}\pi\rho_0 r^3 .$$

The total mass, of course, is $M = \tfrac{4}{3}\pi\rho_0 R^3$.

(b) Rearrange the TOV equation so that the left side has only functions of *p* in it, and the right side has only *r* in it. It will look like $f(p)dp = g(r)dr$.

We start by substituting and rearranging the equation, so we have

$$\frac{dp}{dr} = -\frac{G\left(\tfrac{4}{3}\pi\rho_0 r^3 + 4\pi pr^3\right)(\rho_0+p)}{r\left(r-\tfrac{8}{3}\pi G\rho_0 r^3\right)},$$

$$\frac{dp}{(p+\rho_0)\left(p+\tfrac{1}{3}\rho_0\right)} = \frac{-4\pi Gr^3 dr}{r^2 - \tfrac{8}{3}\pi G\rho_0 r^4},$$

$$\frac{dp}{(p+\rho_0)\left(p+\tfrac{1}{3}\rho_0\right)} = \frac{-4\pi Gr dr}{1-\tfrac{8}{3}\pi G\rho_0 r^2}.$$

(c) Integrate equation (b) to get a relationship between *p* and *r*. Don't forget the constant of integration! The constant of integration will be chosen here or in part (d) so that the pressure at *r* = *R* is zero.

As nasty as this looks, we can integrate it by hand, or we can get a little help from Maple or similar tools. I find it easiest to multiply both sides by $\tfrac{2}{3}\rho_0$, because the left side can now be cleanly separated into two terms that are easy to integrate:

$$\frac{\frac{2}{3}\rho_0 dp}{\left(p+\rho_0\right)\left(p+\frac{1}{3}\rho_0\right)} = \frac{-\frac{8}{3}\pi G\rho_0 r dr}{1-\frac{8}{3}\pi G\rho_0 r^2},$$

$$\int\left(\frac{dp}{p+\frac{1}{3}\rho_0} - \frac{dp}{p+\rho_0}\right) = \frac{1}{2}\int\frac{d\left(-\frac{8}{3}\pi G\rho_0 r^2\right)}{1-\frac{8}{3}\pi G\rho_0 r^2},$$

$$\ln\left(p+\tfrac{1}{3}\rho_0\right) - \ln\left(p+\rho_0\right) = \tfrac{1}{2}\ln\left(1-\tfrac{8}{3}\pi G\rho_0 r^2\right) + k.$$

(d) Do some work to solve the result of eq. (c) for the pressure *p* as a function of *r*. The terms with *G* in them can be simplified by eliminating ρ_0 in favor of the total mass *M* and the total radius *R*.

We exponentiate both sides to yield

$$\frac{p+\frac{1}{3}\rho_0}{p+\rho_0} = e^k\sqrt{1-\tfrac{8}{3}\pi G\rho_0 r^2}.$$

The constant k is chosen so that at $r = R$, the pressure vanishes, and the left side becomes 1/3. We therefore rewrite this as

$$\frac{p+\frac{1}{3}\rho_0}{p+\rho_0} = \frac{\sqrt{1-\frac{8}{3}\pi G\rho_0 r^2}}{3\sqrt{1-\frac{8}{3}\pi G\rho_0 R^2}}.$$

We recall that $M = \frac{4}{3}\pi R^3$, so this formula becomes

$$\frac{p+\frac{1}{3}\rho_0}{p+\rho_0} = \frac{\sqrt{1-2GMr^2/R^3}}{3\sqrt{1-2GM/R}} = \frac{\sqrt{R^3-2GMr^2}}{3R\sqrt{R-2GM}}.$$

We now cross-multiply and solve for p:

$$\left(3p+\rho_0\right)R\sqrt{R-2GM} = \left(p+\rho_0\right)\sqrt{R^3-2GMr^2},$$

$$p\left(3R\sqrt{R-2GM} - \sqrt{R^3-2GMr^2}\right) = \rho_0\left(\sqrt{R^3-2GMr^2} - R\sqrt{R-2GM}\right),$$

$$\frac{p}{\rho_0} = \frac{\sqrt{R^3-2GMr^2} - R\sqrt{R-2GM}}{3R\sqrt{R-2GM} - \sqrt{R^3-2GMr^2}}.$$

(e) The pressure should be highest at the center. Write the pressure at this point. Find the largest radius *R* for fixed *M* such that the pressure is finite at the origin, $p(0) < \infty$.

The pressure at the center is

$$\frac{p}{\rho_0} = \frac{R\sqrt{R} - R\sqrt{R-2GM}}{3R\sqrt{R-2GM} - R\sqrt{R}} = \frac{\sqrt{R} - \sqrt{R-2GM}}{3\sqrt{R-2GM} - \sqrt{R}}.$$

Demanding that this be finite means that the denominator is positive, so

$$3\sqrt{R-2GM}-\sqrt{R}>0,$$
$$9\left(R-2GM\right)>R.$$
$$8R>18GM,$$
$$R>\tfrac{9}{4}GM.$$