Physics 780 – General Relativity

Solution Set Q

41. The Reissner-Nordström solution has many properties similar to the Schwarzschild solution, and therefore many of the same techniques can be used.

(a) Based on the Killing Vectors ∂_t and ∂_ϕ, argue that two components of the four-velocity must be conserved. For consistency with Schwarzschild, call these $-E$ and *J*.

In general, for a Killing vector *K*, the quantity $K^\alpha U_\alpha$ will be conserved, so in this case $U_t = -E$ and $U_\phi = J$ will be conserved.

(b) Argue in a manner similar to Schwarzschild, that if the instantaneous velocity satisfied $U^\theta = U^\phi = 0$ at any time, that these will continue to be true.

We can use formula from the online file SSSST, since the metric is still static, spherically symmetric spacetime. The geodesic equations for these two components of the four-velocity are

$$\frac{d}{d\tau}U^\theta + \Gamma^\theta_{\mu\nu}U^\mu U^\nu = 0 \quad \text{and} \quad \frac{d}{d\tau}U^\phi + \Gamma^\phi_{\mu\nu}U^\mu U^\nu = 0$$

However, the only non-zero components of $\Gamma^\theta_{\mu\nu}$ and $\Gamma^\phi_{\mu\nu}$ have either θ or ϕ or both as lower indices, so the derivatives will vanish and they will simply remain at zero indefinitely.

(c) Argue in a manner similar to Schwarzschild that if at any time $\theta = \frac{1}{2}\pi$ and $U^\theta = 0$, these will continue to be true.

The geodesic equation for U^θ is

$$0 = \frac{d}{d\tau}U^\theta + \Gamma^\theta_{\mu\nu}U^\mu U^\nu = \frac{d}{d\tau}U^\theta + 2\Gamma^\theta_{\theta r}U^\theta U^r + \Gamma^\theta_{\phi\phi}U^\phi U^\phi = \frac{d}{d\tau}U^\theta + 2\Gamma^\theta_{\theta r}U^\theta U^r - \sin\theta\cos\theta U^\phi U^\phi .$$

However, since we have $U^\theta = 0$ and $\sin\theta\cos\theta = 0$, these terms cancel, and hence U^θ will have no derivative and remain at $\theta = \frac{1}{2}\pi$.

(d) For Schwarzschild, we showed that for a massive particle moving in the $\theta = \frac{1}{2}\pi$ plane,

$$\tfrac{1}{2}\left(E^2 - 1\right) = \tfrac{1}{2}\left(U^r\right)^2 + V_{eff}(r) \quad \text{where} \quad V_{eff}(r) = -\frac{GM}{r} + \frac{J^2}{2r^2} - \frac{GMJ^2}{r^3}.$$

Is there a comparable equation for Reissner-Nordström, and if so, what changes must be made to these equations?

If we call $f(r) = -g_{tt} = \left(g_{rr}\right)^{-1}$, then the condition $-1 = U^\alpha U_\alpha$ becomes

$$-1 = U^\alpha U_\alpha = g^{tt}\left(U_t\right)^2 + g_{rr}\left(U^r\right)^2 + g^{\phi\phi}\left(U_\phi\right)^2 = -\frac{1}{f(r)}E^2 + \frac{1}{f(r)}\left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{r^2\sin^2\theta}.$$

Substituting $\theta = \frac{1}{2}\pi$ and multiplying by $f(r)$, we find

$$E^2 - f(r) = \left(\frac{dr}{d\tau}\right)^2 + \frac{J^2}{r^2} f(r)$$

Dividing by two and substituting the explicit form $f(r) = 1 - 2GM/r + GQ^2/r^2$, this becomes

$$\frac{1}{2}\left(E^2 - 1\right) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{GQ^2}{2r^2} + \frac{J^2}{2r^2}\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right).$$

This is identical to what we had before, but with an effective potential

$$V_{eff}(r) = -\frac{GM}{r} + \frac{GQ^2 + J^2}{2r^2} - \frac{J^2 GM}{r^3} + \frac{J^2 GQ^2}{2r^4}.$$

This reduces to the usual formula if $Q = 0$.

42. Consider an object falling into a charged black hole $Q \neq 0$ in the $\theta = \frac{1}{2}\pi$ plane

(a) Show that no matter the value of the energy *E*, it will *never* reach the origin; that is, it will stop moving inwards ($U^r = 0$) before it reaches the center. I had to prove it using two cases, $J = 0$ and $J \neq 0$.

To continue moving all the way to the origin, we must have $U^r \neq 0$, which implies that $\frac{1}{2}\left(E^2 - 1\right) - V_{eff}(r) > 0$. But if you look at small *r*, it is obvious that if $J \neq 0$, the last term dominates and runs to infinity, which makes it impossible to satisfy this inequality. Also, if $J = 0$, the second term dominates, and since this is also positive and divergent, it is again impossible to satisfy $\frac{1}{2}\left(E^2 - 1\right) - V_{eff}(r) > 0$. So it cannot reach the origin.

(b) Consider now the case of dropping something in from infinity, initially at rest, so $E = 1$ and $J = 0$. Find the turning radius $r_{\min}$ where it stops falling and makes a U-turn. Where does this point compare to the two event horizons, $r_\pm$ (assume we don't have a naked singularity, so $Q < M\sqrt{G}$)?

The particle will stop when the energy term $\frac{1}{2}\left(E^2 - 1\right) = V_{eff}(r)$, so we need $V_{eff}(r) = 0$. Setting $J = 0$, this happens when

$$0 = -\frac{GM}{r} + \frac{GQ^2}{2r^2}, \quad r_{\min} = \frac{Q^2}{2M}.$$

We note in the limit of small Q, this is smaller than $r_\pm$, and we therefore speculate that this is always true. To test it, we write the inequality and then transform it into a true equation, namely

$$\frac{Q^2}{2M} < GM - \sqrt{G^2M^2 - GQ^2}\,,$$
$$\sqrt{G^2M^2 - GQ^2} < GM - \frac{Q^2}{2M}$$
$$G^2M^2 - GQ^2 < \left(GM - \frac{Q^2}{2M}\right)^2 = G^2M^2 - GQ^2 + \frac{Q^4}{4M^2}.$$

The final equation is obviously true. On the middle step, we are allowed to square it since it's easy to show both sides are positive given that $Q < M\sqrt{G}$.

(c) Find a formula for the proper time to fall from distance *r* to $r_{\min}$.

We set $E = 1$ and $J = 0$ and solve for $U^r = dr/d\tau$, so we have

$$0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{GQ^2}{2r^2},$$
$$\frac{dr}{d\tau} = \sqrt{\frac{2GM}{r} - \frac{GQ^2}{r^2}} = \sqrt{\frac{2GM}{r}\left(1 - \frac{r_{\min}}{r}\right)}.$$

We now find the proper time by integrating

$$\tau = \int d\tau = \int dr\left(\frac{dr}{d\tau}\right) = \int \frac{rdr}{\sqrt{2GM\left(r - r_{\min}\right)}}$$

To perform the integral, set $x = r - r_{\min}$, then we have

$$\tau = \frac{1}{\sqrt{2GM}}\int \frac{\left(x + r_{\min}\right)dx}{\sqrt{x}} = \frac{1}{\sqrt{2GM}}\int\left(x^{1/2} + r_{\min}x^{-1/2}\right)dx$$
$$= \frac{1}{\sqrt{2GM}}\left(\frac{2}{3}x^{3/2} + 2r_{\min}x^{1/2}\right) = \frac{1}{\sqrt{2GM}}\left[\frac{2}{3}\left(r - r_{\min}\right)^{3/2} + 2r_{\min}\left(r - r_{\min}\right)^{1/2}\right]$$
$$= \sqrt{\frac{r - r_{\min}}{2GM}}\left(\frac{2}{3}r - \frac{2}{3}r_{\min} + 2r_{\min}\right) = \frac{2}{3}\left(r + 2r_{\min}\right)\sqrt{\frac{r - r_{\min}}{2GM}} = \frac{2}{3}\left(r + \frac{Q^2}{M}\right)\sqrt{\frac{r - Q^2/2M}{2GM}}$$
$$= \frac{\left(Mr + Q^2\right)\sqrt{2Mr - Q^2}}{3M^2\sqrt{G}}.$$

I don't have anything particularly inciteful to say about this final answer. It's a bit of a mess.