Physics 780 – General Relativity

Solution Set R

43. [15] This problem has a lot to do with units. The goal is to keep careful track of them.

(a) Working in SI units, if you have a charge *q*, what is the electric field at a distance *r*? Compare with our formula for the electric field for the Reissner-Nordström. At least at large *r*, they should be the same. Based on this, find a formula relating *Q* to *q*.

In SI units, the electric field from a point charge q is $q/4\pi\varepsilon_0 r^2$. We would expect, at least at large r, for this to match the electric field we have. Matching the two formulas, we have

$$\frac{q}{4\pi\varepsilon_0 r^2} = \frac{Q}{\sqrt{4\pi\varepsilon_0}\, r^2} \quad \text{so} \quad Q = \frac{q}{\sqrt{4\pi\varepsilon_0}}.$$

The formula works at all r, which we wouldn't have expected.

(b) You're not done with units! Because we are working in general relativity, there can easily be some factors of *c*, the speed of light hidden in your formula for part (a). Given that GQ^2 has units of m², revise your formula from part (a) by adding an appropriate power of *c* to the relationship you found there.

Naively, keeping track *only* of units, we would naively have

$$GQ^2 = \frac{Gq^2}{4\pi\varepsilon_0} \sim \frac{\left(\mathrm{m^3kg^{-1}s^{-2}}\right)\cdot \mathrm{C^2}}{\mathrm{m^{-3}kg^{-1}s^2C^2}} \sim \frac{\mathrm{m^6}}{\mathrm{s^4}}.$$

This is not units of $\mathrm{m^2}$, but we can make it so by dividing by c^4, so we must have

$$GQ^2 = \frac{Gq^2}{4\pi\varepsilon_0 c^4} \sim \frac{\left(\mathrm{m^3kg^{-1}s^{-2}}\right)\cdot \mathrm{C^2}}{\mathrm{m^{-3}kg^{-1}s^2C^2}\cdot\left(\mathrm{m/s}\right)^4} \sim \mathrm{m^2}.$$

Converting backwards, we now have

$$Q = \frac{q}{c^2\sqrt{4\pi\varepsilon_0}}.$$

(c) At large distances, we can use classical formulas to calculate forces. Supposed a black hole of mass *M* and charge *q* is so charged up that a proton with mass *m* and charge *e* far from the black hole feels exactly balancing forces from gravity and electromagnetism. What is the ratio *q*/*M* for this black hole in C/kg? You can use classical formulas, since we are far from the black hole.

The gravitational force between a black hole and a proton is GMm_p/r^2. The electric force is $qe/4\pi\varepsilon_0 r^2$. We equate these to find

$$\frac{GMm_p}{r^2} = \frac{qe}{4\pi\varepsilon_0 r^2},$$

$$\frac{q}{M} = \frac{4\pi G\varepsilon_0 m_p}{e} = \frac{4\pi\left(6.674\times10^{11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}\right)\left(8.854\times10^{-12}\ \text{m}^{-3}\text{kg}^{-1}\text{s}^2\text{C}^2\right)\left(1.673\times10^{-27}\ \text{kg}\right)}{1.602\times10^{-19}\ \text{C}}$$
$$= 7.752\times10^{-29}\ \text{C/kg}$$

(d) For the black hole in part (c), find the value of $Q\big/\left(M\sqrt{G}\right)$. You may have to include factors of *c* to make this expression dimensionless.

Substituting in from before and not worrying about factors of *c*, we have

$$\frac{Q}{M\sqrt{G}} = \frac{q}{M\sqrt{4\pi\varepsilon_0 G}} = \frac{7.752\times10^{-29}\ \text{C/kg}}{\sqrt{4\pi\left(6.674\times10^{11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}\right)\left(8.854\times10^{-12}\ \text{m}^{-3}\text{kg}^{-1}\text{s}^2\text{C}^2\right)}} = 8.996\times10^{-19}.$$

Remarkably, we didn't have to cancel anything out. Obviously, if you try to charge up your black hole by dropping protons into it, you are going to end up with a minuscule charge compared to your mass.

44. Although the Kerr metric is not diagonal, it *is* diagonal on the *z*-axis, $\theta = 0$. By symmetry, any object moving along the *z*-axis will continue moving along the *z*-axis.

(a) Find the metric and the inverse metric on the *z*-axis as a function of *r*. Ignore the $g_{\phi\phi}$ part of the metric. As a check, you should find that $g^{tt} = -g_{rr}$.

We simply replace $\sin^2\theta = 0$ and $\cos^2\theta = 1$. Then $\rho^2 = r^2 + a^2$, and the metric becomes

$$ds^2 = -\left(1 - \frac{2GMr}{r^2 + a^2}\right)dt^2 + \frac{r^2 + a^2}{r^2 + a^2 - 2GMr}dr^2 + \left(r^2 + a^2\right)d\theta^2.$$

The $d\phi^2$ term disappears from the metric, but we understand that that is just an effect of the bad coordinates at $\theta = 0$. Since we have no motion in the θ or ϕ-direction, it is irrelevant. Since it's diagonal, we can just take the inverse, which gives

$$g^{tt} = -\frac{r^2 + a^2}{r^2 + a^2 - 2GMr},\quad g^{rr} = \frac{r^2 + a^2 - 2GMr}{r^2 + a^2},\quad g^{\theta\theta} = \frac{1}{r^2 + a^2}.$$

(b) As usual, since we have a time translation symmetry, ∂_t, there will be a conserved component of the four-velocity, whose value we will call –*E*. If an object starts at rest from infinity, what is *E*?

Because ∂_t is a Killing vector, $U_t = -E$ will be constant. At infinity, an object at rest will have all the space components $U^i = 0$, and since space-time is asymptotically flat, $U^t = 1$, and therefore $U_t = -E = -1$, so $E = 1$.

(c) By demanding that $U^\mu U_\mu = -1$, find a formula for U^r as a function of *r* for an object that starts at rest at infinity. Note that we are assuming $U^\phi = U^\theta = 0$.

Keeping in mind that the metric is diagonal, we have

$$-1 = g^{tt}U_t U_t + g_{rr}U^r U^r = \frac{r^2 + a^2}{r^2 + a^2 - 2GMr}\left[\left(U^r\right)^2 - 1\right],$$

$$\left(U^r\right)^2 = 1 - \frac{r^2 + a^2 - 2GMr}{r^2 + a^2} = \frac{2GMr}{r^2 + a^2},$$

$$U^r = \frac{dr}{d\tau} = \sqrt{\frac{2GMr}{r^2 + a^2}}.$$

(d) Find a formula for the time it takes to fall from a distance *r* to *r* = 0. It will be an integral that you probably can't do.

We simply write the integral as

$$\tau = \int d\tau = \int dr\left(\frac{dr}{d\tau}\right)^{-1} = \int dr\sqrt{\frac{r^2 + a^2}{2GMr}}.$$

I was unable to do this integral, though Maple gave me some incomprehensible expression in terms of incomplete elliptic integrals.

(e) Even if you can't do the integral, convince yourself that it is finite. I did it by setting $r = ax^2$, and then convincing myself that the resulting integral was finite for any finite *r*.

If we substitute $r = ax^2$, we find

$$\tau = \int 2ax\,dx\sqrt{\frac{x^4 a^2 + a^2}{2GMax^2}} = \sqrt{\frac{2a^3}{GM}}\int_0^{\sqrt{r/a}} dx\sqrt{x^4 + 1}.$$

Though I can't to this integral, the integrand is clearly finite, so we are done.