Physics 780 – General Relativity

Solution Set S

45. Imagine we have an empty universe, so $\rho = 0$.

(a) Using the first Friedmann equation, what must be the value of *k*? Solve for *a*(*t*) as a function of time, choosing the constant of integration so that *a*(0) = 0.

The first Friedmann equation is just

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho - \frac{k}{a^2}$$

Since the density is zero, the first term on the right vanishes. Since the left side is positive, the remaining term must be positive, and since *k* can only take the values $k \in \{0, \pm 1\}$, we must have $k = -1$. Substituting in, we have

$$\frac{\dot{a}^2}{a^2} = \frac{1}{a^2} \quad \Rightarrow \quad \dot{a}^2 = 1 \quad \Rightarrow \quad \frac{da}{dt} = 1.$$

Integrating this last equation, and choosing the constant of integration as suggested, we have $a = t$.

(b) Write the full metric. We have just discovered a new metric, different from flat space, with nothing in it! Or have we? Big hint: look at problem set F, problem 16.

The full metric is just

$$ds^2 = -\mathrm{d}t^2 + t^2 \left[\frac{\mathrm{d}r^2}{1+r^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) \right].$$

As demonstrated in problem 16, this is really just flat spacetime disguised.

46. Suppose the universe is flat (*k* = 0) and is filled with a fluid of just one type with $\rho \propto a^{-n}$, with *n* > 0. I recommend writing $\frac{8}{3}\pi G\rho = Ca^{-n}$, where *C* is constant.

(a) Using the first Friedmann equation, write a formula of the form $dt = f(a)\,da$, where $f(a)$ is a simple formula. Integrate it to get a formula for the age of the universe *t* in terms of *a*, defining *t* = 0 as the time when *a* = 0.

Given that *k* = 0, the first Friedmann equation says that

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho = Ca^{-n},$$

$$a^{n-2}\left(\frac{da}{dt}\right)^2 = C,$$

$$\frac{1}{\sqrt{C}} a^{n/2-1} da = dt.$$

We now simply integrate this equation to get

$$t = \int \frac{1}{\sqrt{C}} a^{n/2-1} da = \frac{2}{n} a^{n/2} \frac{1}{\sqrt{C}}.$$

(b) Using the fact that H_0 is the current value of $\dot{a}/a$, find a formula for the current age of the universe *just* in terms of H_0 and n.

Substituting into the Friedmann equation today, we see that

$$H_0^2 = \frac{8\pi}{3} G\rho_0 = Ca_0^{-n/2},$$

$$H_0 = \sqrt{C} a_0^{-n/2}.$$

We note that the combination we need to find the current age of the universe, which is thefore

$$t_0 = \frac{2}{nH_0}.$$

(c) The current value of the age of Hubble's constant is H_0 = 67.7 km/s/Mpc. Find the value of H_0^{-1}, called the *Hubble time*, in Gyr.

A Mpc is a million parsecs, and we can look up the units, so we have

$$H_0^{-1} = \frac{\text{s} \cdot \text{Mpc}}{67.7 \text{ km}} \cdot \frac{\text{km}}{10^3 \text{ m}} \cdot \frac{10^6 \text{ pc}}{\text{Mpc}} \cdot \frac{3.086 \times 10^{16} \text{ m}}{\text{pc}} \cdot \frac{1 \text{ yr}}{3.156 \times 10^7 \text{ s}} = 1.44 \times 10^{10} \text{ yr} = 14.4 \text{ Gyr}.$$

(d) Assuming we have matter (n = 3) or radiation (n = 4), based on parts (b) and (c) how old is the universe in each case? Compare to the age of the oldest stars, somewhere around 13 Gyr.

We simply multiply by the factor $\frac{2}{3}$ for matter or $\frac{2}{4} = \frac{1}{2}$ for radiation to yield a total age of 9.60 Gyr for matter or 7.20 Gyr for radiation. Both are substantially less than the age of the oldest stars, around 13 Gyr.