Physics 780 – General Relativity

Solution Set U

49. Way back in the previous millennium (i.e., pre-1995) we only knew about matter (and a tiny bit of radiation), and were none too confident about the value of Ω_m. For *this* problem, assume the universe contains matter only.

(a) Show that if $\Omega_m \le 1$, the universe will never stop growing, *i.e.*, there is no time in the future when $\dot{a} = 0$.

(b) Show that if $\Omega_m > 1$, it is inevitable that the universe will eventually stop growing. Find a formula for the size of the universe compared to now, a/a_0, when the universe will stop growing as a function of Ω_m.

We start with the first Friedmann equation, which says

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho - \frac{k}{a^2}$$

Evaluating this today, and recalling that $H^2\Omega_m = \frac{8}{3}\pi G\rho_m$, we have

$$H_0^2 = H_0^2\Omega_{m0} - \frac{k}{a_0^2}.$$

Substituting it is easy to solve this to show that $-k/a_0^2 = H_0^2\left(1-\Omega_{m0}\right)$. Keeping in mind that $\rho \propto a^{-3}$ and obviously $-k/a^2 \propto a^{-2}$, we conclude that

$$\frac{8\pi}{3}G\rho_m = \frac{8\pi}{3}G\rho_{m0}\left(\frac{a_0^3}{a^3}\right) = H_0^2\Omega_m\frac{a_0^3}{a^3} \quad \text{and} \quad -\frac{k}{a^2} = -\frac{k}{a_0^2}\frac{a_0^2}{a^2} = H_0^2\left(1-\Omega_{m0}\right)\frac{a_0^2}{a^2}.$$

We now substitute this into the Friedmann equation, and we have

$$\frac{\dot{a}^2}{a^2} = H_0^2\Omega_{m0}\frac{a_0^3}{a^3} + H_0^2\left(1-\Omega_{m0}\right)\frac{a_0^2}{a^2}$$

We see that if $\Omega_m \le 1$, both terms on the right are positive, and hence one can never have $\dot{a} = 0$. In contrast, if $\Omega_m > 1$, then as *a* grows the term on the right will ultimately win out, and the universe will stop growing when the two terms cancel, which means

$$0 = H_0^2\Omega_{m0}\frac{a_0^3}{a^3} + H_0^2\left(1-\Omega_{m0}\right)\frac{a_0^2}{a^2},$$

$$\Omega_{m0}\frac{a_0^3}{a^3} = \left(\Omega_{m0}-1\right)\frac{a_0^2}{a^2},$$

$$\frac{a}{a_0} = \frac{\Omega_{m0}}{\Omega_{m0}-1}$$

50. Suppose in some gauge choice, an almost-flat universe has metric perturbations that satisfy the harmonic condition, $\partial_\mu h^\mu{}_\beta(x) = \frac{1}{2}\partial_\beta h^\mu{}_\mu(x)$.

(a) Show that if we make a small coordinate change $x^\mu \to x'^\mu + \xi^\mu$, the harmonic condition is still preserved if $\Box\xi_\mu = 0$.

We recall that under this condition, the metric perturbation h changes to $h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$. We therefore are trying to prove that

$$\partial_\mu h'^\mu{}_\beta = \tfrac{1}{2}\partial_\beta h'^\mu{}_\mu ,$$
$$\partial_\mu h^\mu{}_\beta - \eta^{\mu\alpha}\left(\partial_\mu\partial_\alpha\xi_\beta + \partial_\mu\partial_\beta\xi_\alpha\right) = \tfrac{1}{2}\partial_\beta\left(h^\mu{}_\mu - \eta^{\mu\alpha}\partial_\alpha\xi_\mu - \eta^{\mu\alpha}\partial_\mu\xi_\alpha\right),$$
$$\partial_\mu h^\mu{}_\beta - \Box\xi_\beta - \eta^{\mu\alpha}\partial_\mu\partial_\beta\xi_\alpha = \tfrac{1}{2}\partial_\beta h^\mu{}_\mu - \eta^{\mu\alpha}\partial_\beta\partial_\mu\xi_\alpha$$

The first term on each side is valid by assumption, the last terms are identical, and the middle term on the left is what is required to make this statement true.

(b) Suppose we are looking at wave solutions $h_{\mu\nu}(x) = h_{\mu\nu}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$, with $|\mathbf{k}| = \omega$. Show that the coordinate change $\xi_0 = (ih_{00}/2\omega)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$ will cause $h'_{00} = 0$.

This coordinate change causes h' to become

$$h'_{00}(x) = h_{00}(x) - \partial_0\xi_0(x) - \partial_0\xi_0(x) = h_{00}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} - (-2i\omega)(ih_{00}/2\omega)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$$
$$= (h_{00} - h_{00})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} = 0.$$

This coordinate change will also cause h_{i0} to change, but since we haven't specified it yet.

(c) Show that a subsequent coordinate change $\xi_i = (ih_{i0}/\omega)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$ will cause $h'_{i0} = 0$.

This coordinate change causes h' to become

$$h'_{i0}(x) = h_{i0}(x) - \partial_i\xi_0(x) - \partial_0\xi_i(x) = h_{i0}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} - 0 - (-i\omega)(ih_{i0}/\omega)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$$
$$= (h_{i0} - h_{i0})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} = 0.$$

Hence we can get rid of all the terms of the form $h_{\mu 0} = h_{0\mu}$.

51. We defined gravity waves by writing $h_{\mu\nu}(x) = h_{\mu\nu}\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$, and then writing *h* in terms of two polarization vectors $h_{\mu\nu} = h_+ e^+_{\mu\nu} + h_\times e^\times_{\mu\nu}$. These are not the *only* choices for the basis tensors for gravity waves. Let's assume k is in the *z*-direction.

(a) Define the right-helicity and left-helicity vectors as $e^R_{\mu\nu} = e^+_{\mu\nu} + ie^\times_{\mu\nu}$ and $e^L_{\mu\nu} = e^+_{\mu\nu} - ie^\times_{\mu\nu}$. Show that any wave can be written in the form $h_{\mu\nu} = h_R e^R_{\mu\nu} + h_L e^L_{\mu\nu}$, and find formulas for $h_{R,L}$ in terms of $h_{+,\times}$ and vice-versa.

We start by writing

$$h_{\mu\nu} = h_R e^R_{\mu\nu} + h_L e^L_{\mu\nu} = h_R\left(e^+_{\mu\nu} + ie^\times_{\mu\nu}\right) + h_L\left(e^+_{\mu\nu} - ie^\times_{\mu\nu}\right) = \left(h_R + h_L\right)e^+_{\mu\nu} + i\left(h_R - h_L\right)e^\times_{\mu\nu}.$$

Matching to $h_{\mu\nu} = h_+ e^+_{\mu\nu} + h_\times e^\times_{\mu\nu}$, we see that

$$h_+ = h_R + h_L \quad \text{and} \quad h_\times = ih_R - ih_L .$$

It is easy to invert these equations and find expressions for h_R and h_L, namely

$$h_R = \tfrac{1}{2}\left(h_+ - ih_\times\right) \quad \text{and} \quad h_L = \tfrac{1}{2}\left(h_+ + ih_\times\right).$$

(b) Consider a wave containing only the right-helicity wave (*i.e.* $h_L = 0$). Perform a rotation of this wave around the *z*-axis by an angle θ, with inverse Lorentz transformation

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that such a transformation simply multiplies the metric perturbation $h_{\mu\nu}$ by a phase $e^{im\theta}$ and determine the value of the *helicity m*. Note that Lorentz transforms on indices that are down work as $h'_{\mu\nu} = h_{\alpha\beta}\left(\Lambda^{-1}\right)^\alpha{}_\mu\left(\Lambda^{-1}\right)^\beta{}_\nu$. Repeat for the left-helicity wave.

The right-helicity wave would have the form

$$h_{ij}(x) = h_R\left(e^+_{ij} + ie^\times_{ij}\right)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} = h_R\left(e^+_{ij} + ie^\times_{ij}\right)e^{i\omega z - i\omega t}.$$

The only non-zero components of this are

$$h_{11} = h_R, \quad h_{22} = -h_R, \quad h_{12} = h_{21} = ih_R .$$

The rotation has no effect on *z* or *t*, so the only effect is to rotate the components of *h*, so

$$
\begin{aligned}
h'_{11} &= h_{11}\cos^2\theta + h_{12}\cos\theta\sin\theta + h_{21}\sin\theta\cos\theta + h_{22}\sin^2\theta \\
&= h_R\left(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta\right) = h_R e^{2i\theta} = h_{11}e^{2i\theta}, \\
h'_{12} &= h_{11}\cos\theta\left(-\sin\theta\right) + h_{12}\cos^2\theta + h_{21}\sin\theta\left(-\sin\theta\right) + h_{22}\sin\theta\cos\theta \\
&= h_R\left(-\cos\theta\sin\theta + i\cos^2\theta - i\sin^2\theta - \sin\theta\cos\theta\right) = ih_R e^{2i\theta} = h_{12}e^{2i\theta}, \\
h'_{21} &= h_{11}\left(-\sin\theta\right)\cos\theta + h_{12}\left(-\sin\theta\right)\sin\theta + h_{21}\cos^2\theta + h_{22}\cos\theta\sin\theta \\
&= h_R\left(-\sin\theta\cos\theta - i\sin^2\theta + i\cos^2\theta - \sin\theta\cos\theta\right) = ih_R e^{2i\theta} = h_{12}e^{2i\theta}, \\
h'_{22} &= h_{11}\left(-\sin\theta\right)\left(-\sin\theta\right) + h_{12}\left(-\sin\theta\right)\cos\theta + h_{21}\cos\theta\left(-\sin\theta\right) + h_{22}\cos^2\theta \\
&= h_R\left(\sin^2\theta - i\sin\theta\cos\theta - i\sin\theta\cos\theta - \cos^2\theta\right) = -h_R e^{2i\theta} = h_{22}e^{2i\theta}.
\end{aligned}
$$

That was painful. We now need to repeat this for the left-handed variety, for which

$$
h_{11} = h_L, \quad h_{22} = -h_L, \quad h_{12} = h_{21} = -ih_L.
$$

Substituting this into our expressions, we have

$$
\begin{aligned}
h'_{11} &= h_{11}\cos^2\theta + h_{12}\cos\theta\sin\theta + h_{21}\sin\theta\cos\theta + h_{22}\sin^2\theta \\
&= h_L\left(\cos^2\theta - 2i\cos\theta\sin\theta - \sin^2\theta\right) = h_L e^{-2i\theta} = h_{11}e^{-2i\theta}, \\
h'_{12} &= h_{11}\cos\theta\left(-\sin\theta\right) + h_{12}\cos^2\theta + h_{21}\sin\theta\left(-\sin\theta\right) + h_{22}\sin\theta\cos\theta \\
&= h_L\left(-\cos\theta\sin\theta - i\cos^2\theta + i\sin^2\theta - \sin\theta\cos\theta\right) = -ih_L e^{-2i\theta} = h_{12}e^{-2i\theta}, \\
h'_{21} &= h_{11}\left(-\sin\theta\right)\cos\theta + h_{12}\left(-\sin\theta\right)\sin\theta + h_{21}\cos^2\theta + h_{22}\cos\theta\sin\theta \\
&= h_R\left(-\sin\theta\cos\theta + i\sin^2\theta - i\cos^2\theta - \sin\theta\cos\theta\right) = -ih_L e^{-2i\theta} = h_{12}e^{-2i\theta}, \\
h'_{22} &= h_{11}\left(-\sin\theta\right)\left(-\sin\theta\right) + h_{12}\left(-\sin\theta\right)\cos\theta + h_{21}\cos\theta\left(-\sin\theta\right) + h_{22}\cos^2\theta \\
&= h_R\left(\sin^2\theta + i\sin\theta\cos\theta + i\sin\theta\cos\theta - \cos^2\theta\right) = -h_L e^{-2i\theta} = h_{22}e^{-2i\theta}.
\end{aligned}
$$

It’s clear that the helicity is $m = +2$ for right-handed waves, and $m = –2$ for left-handed waves.