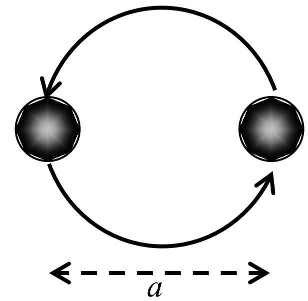
Physics 780 – General Relativity

Solution Set X



56. [25] Suppose we have two non-relativistic objects of mass *M* orbiting each other in circular orbits with a separation *a* (so the radius of the orbit is *a*/2).

(a) Since the objects are non-relativistic, we can use Newtonian approximations. By equating the gravitational force to the centripetal force $F = M\omega^2(a/2)$, find a formula for the angular velocity for the orbit ω.

The gravitational force between the two objects is GM^2/a^2. The centripetal force required to keep them moving in a circle of radius $r = a/2$ is $M\omega^2(a/2)$. Equating these two quantities, we have

$$\frac{GM^2}{a^2} = \frac{M\omega^2 a}{2},$$
$$\omega^2 = \frac{2GM}{a^3}$$

(b) Write the position of each particle, assuming they are orbiting in the *xy*-plane about the origin, as a function of time.

The positions of the particles can be written as $x = \pm\frac{1}{2}a\cos(\omega t)$ and $y = \pm\frac{1}{2}a\sin(\omega t)$, with the ± denoting which of the two particles we are talking about.

(c) Find the moments $Q_{ij} = \sum_a m_a x_a^i x_a^j$ as a function of time.

We have

$$Q_{xx} = m_1 x_1^2 + m_2 x_2^2 = M\left[\tfrac{1}{2}a\cos(\omega t)\right]^2 + M\left[-\tfrac{1}{2}a\cos(\omega t)\right]^2 = \tfrac{1}{2}Ma^2\cos^2(\omega t),$$
$$Q_{yy} = m_1 y_1^2 + m_2 y_2^2 = M\left[\tfrac{1}{2}a\sin(\omega t)\right]^2 + M\left[-\tfrac{1}{2}a\sin(\omega t)\right]^2 = \tfrac{1}{2}Ma^2\sin^2(\omega t),$$
$$Q_{xy} = Q_{yx} = m_1 x_1 y_1 + m_2 x_2 y_2 = M\left[\tfrac{1}{2}a\cos(\omega t)\right]\left[\tfrac{1}{2}a\sin(\omega t)\right] + M\left[-\tfrac{1}{2}a\cos(\omega t)\right]\left[-\tfrac{1}{2}a\sin(\omega t)\right]$$
$$= \tfrac{1}{2}Ma^2\cos(\omega t)\sin(\omega t).$$

(d) Rewrite $Q_{ij}(t)$ as a constant term plus oscillatory terms. What is the frequency of the oscillatory terms? Write the oscillating terms as $Q_{ij}(t) = (\text{constant}) + Q_{ij}e^{-i\omega t} + Q_{ij}^{*}e^{i\omega t}$.

We can write the moments using double-angle formulas as

$$Q_{xx} = \tfrac{1}{4}Ma^2\left[1+\cos(2\omega t)\right] = \tfrac{1}{4}Ma^2 + \tfrac{1}{8}Ma^2e^{-2i\omega t} + \tfrac{1}{8}Ma^2e^{2i\omega t},$$
$$Q_{yy} = \tfrac{1}{4}Ma^2\left[1-\cos(2\omega t)\right] = \tfrac{1}{4}Ma^2 - \tfrac{1}{8}Ma^2e^{-2i\omega t} - \tfrac{1}{8}Ma^2e^{2i\omega t},$$
$$Q_{xy} = Q_{yx} = \tfrac{1}{4}Ma^2\sin(2\omega t) = \tfrac{i}{8}Ma^2e^{-2i\omega t} - \tfrac{i}{8}Ma^2e^{2i\omega t}.$$

The non-oscillating coefficients are then $Q_{xx} = \tfrac{1}{8}Ma^2$, $Q_{yy} = -\tfrac{1}{8}Ma^2$, and $Q_{xy} = Q_{yx} = \tfrac{1}{8}iMa^2$. Note that the frequency is 2ω.

(e) Find the power radiated $P = \tfrac{2}{5}G\omega^6c^{-5}Q_{ij}^{*}Q_{ij}$.

The formula should have $Q_{ij}^{*}Q_{ij} - \tfrac{1}{3}\left|Q_{ii}\right|^2$, but the trace in this case is zero, so it doesn't contribute. Putting everything together, we have

$$P = \tfrac{2}{5}G\omega_r^6c^{-5}\left(Q_{ij}^{*}Q_{ij} - \tfrac{1}{3}Q_{ii}^{*}Q_{jj}\right) = \tfrac{2}{5}G(2\omega)^6c^{-5}\left[\left(\tfrac{1}{8}Ma^2\right)^2 + \left(\tfrac{1}{8}Ma^2\right)^2 + \left(\tfrac{1}{8}Ma^2\right)^2 + \left(\tfrac{1}{8}Ma^2\right)^2\right]$$
$$= \tfrac{2}{5}\cdot 64\cdot\tfrac{1}{64}\cdot 4G\omega^6M^2a^4c^{-5} = \tfrac{8}{5}G\omega^6M^2a^4c^{-5}$$

We now substitute the equation from part (a) to eliminate ω, to yield

$$P = \frac{8}{5}G\left(\frac{2GM}{a^3}\right)^3M^2a^4c^{-5} = \frac{64G^4M^5}{5c^5a^5}.$$

(f) Find total energy $E = K + V$, where the potential energy is $V = -GM^2/a$, and *K* is the sum of the two potential energies, each of which is $K_a = \tfrac{1}{2}M(\omega a/2)^2$. You should find that the potential energy is exactly twice as big as the kinetic term (and of opposite sign).

This is straightforward, so

$$E = K + V = 2\cdot\frac{1}{2}Mv^2 - \frac{GM^2}{a} = M\left(\frac{\omega a}{2}\right)^2 - \frac{GM^2}{a} = \frac{Ma^2}{4}\cdot\frac{2GM}{a^3}$$
$$= \frac{GM^2}{2a} - \frac{GM^2}{a} = -\frac{GM^2}{2a}.$$

(g) Find a formula for the characteristic time $\tau = |E|/P$ it will take for the orbit to decay to radius zero. Evaluate it for $M = M_{Sun} = 1.989\times10^{30}\ \text{kg}$ and $a = 2\times10^{4}\ \text{km}$.

Dividing our two formulas, we have

$$\tau = \frac{|E|}{P} = \frac{GM^2}{2a}\frac{5c^5a^5}{64G^4M^5} = \frac{5c^5a^4}{128G^3M^3}$$
$$= \frac{5\left(2.998\times10^8\ \text{m/s}\right)^5\left(2\times10^7\ \text{m}\right)^4}{128\left(6.674\times10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}\right)^3\left(1.989\times10^{30}\,\text{kg}\right)^3} = 6.47\times10^9\ \text{s} = 205\ \text{years}.$$

A more careful analysis will show that the actual time is one-fourth of this value (because it radiates faster as the two stars spiral towards each other), so it is more like fifty years.