Problems 1-5

1. Define a dot product for 1-forms $\tilde{\omega}^i \cdot \tilde{\omega}^j \equiv g^{ij}$
   (a) Show that $\tilde{u} \cdot \tilde{v} \equiv \langle \tilde{u}, \tilde{v} \rangle$ implies $u^i = g^{ij} u_j$
   (b) Show that if we think of $g^{ij}$ as a matrix, then the matrix $g^{ij} = (g^{ij})^{-1}$.
      hint: $M^{-1} \cdot M = I$

2. Consider a 2d skew Cartesian coordinates, related to conventional coordinates by
   \[ x = \bar{x} + \bar{y} \sin \theta \]
   \[ y = \bar{y} \cos \theta \]

   (a) Express $d\bar{x}^i$ in terms of $d\bar{x}^i$.
   (b) Express $\partial_{\bar{x}}$ in terms of $\partial_\theta$
   (c) Find the components of $g^{ij}$ and $g_{ij}$.
   (d) Find the lengths of $\partial_\theta$ and $\partial_{\bar{y}}$ and the angle between them.
   (e) Draw the vectors $\partial_\theta$, $\partial_{\bar{y}}$, $\partial_x$ and $\partial_y$ on a diagram of the coordinate system at one point.

3. Spherical coordinates are defined by
   \[ x = r \sin \theta \cos \phi \]
   \[ y = r \sin \theta \sin \phi \]
   \[ z = r \cos \theta \]

   (a) Show that $g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$
   (b) What is the volume element $d^3V$ in spherical coordinates?

4. Show that $\nabla_i \alpha_j = \partial_i \alpha_j - \Gamma^k_{ji} \alpha_k$. Use Leibniz’s rule, $\nabla_i (AB) = (\nabla_i A)B + A\nabla_i B$.

5. Use the fact that $\nabla_i v^j$ transforms as a tensor to prove one of the following two relationships: