Problems 25-28

25. Define the first moment of the four momentum of a distribution of matter in flat spacetime as

\[ V^{\alpha\beta} = \int x^{\alpha} T^{\beta\beta} d^3 \vec{x} \]

(a) Show that the time derivative of \( V \) is

\[ \partial_\alpha V^{\alpha\beta} = \int T^{\alpha\beta} d^3 \vec{x} \]

It may be helpful to consider separately the cases \( \alpha = 0 \) and \( \alpha = i \).

(b) From this, show that the angular momentum as defined in class is conserved. Also, show that the dipole moment satisfies

\[ \vec{D}(t) = \vec{D}(0) + \vec{P} t \]

26. Show that for a stationary distribution of matter \( (\partial_\alpha T^{\alpha\beta} = 0) \),

(a) \[ \int T^{ij} d^3 \vec{x} = 0 \]

(b) \[ \int x^{(i} T^{jk)} d^3 \vec{x} = 0 \] (hint: consider the time derivative of \( \int x^{\alpha} T^{\alpha\beta} d^3 \vec{x} \))

27. A two-index tensor \( T^{\mu\nu} \) or \( T_{\mu\nu} \) is diagonal if the only non-zero components have \( \mu = \nu \).

(a) Show that if the metric \( g_{\mu\nu} \) is diagonal, then for any diagonal tensor \( T^{\mu\nu} \) we will have \( T^{\mu\nu} = T^{\nu\mu} \), i.e., when written with one index down and one up, it will be the same thing. What is \( T^{\mu\nu} \) for a perfect fluid at rest?

(b) For a general spherically symmetric time-independent metric,

\[ ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

use conservation of the stress-energy tensor, \( \nabla_\mu T^{\mu\nu} = 0 \) with \( \nu = r \) to show that

\[ \partial_r P = -\frac{1}{r} \left( \rho + P \right) \partial_r \left( \ln f \right) \]

28. In standard coordinates, the Schwarzschild metric takes the form

\[ ds^2 = -(1 - 2GM/r) dt^2 + \left(1 - 2GM/r\right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

Change coordinates to a new radial coordinate \( R \) defined by

\[ r = R + GM + \frac{G^2 M^2}{4R} = \left( R + \frac{1}{2} GM \right)^2 / R \]

Show that in this new coordinate system, the metric can be rewritten as

\[ ds^2 = -A(R) dt^2 + B(R) \left[ dR^2 + R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \]

and determine the new metric functions \( A(R) \) and \( B(R) \).