Problems 29-31

29. An object is dropped from rest at infinity directly into a Schwarzschild metric, which we will assume is valid all the way to \( r = 0 \).
(a) What is the initial “energy” \( E \) and “angular momentum” \( L \)? Calculate a formula for \( u^r = dr/d\tau \) as a function of \( r \). Also, find a formula for \( dr/dt \).
(b) Integrate the equation you found in (a) to find the proper time \( \tau \) it takes to fall from arbitrary \( r \) to the origin.
(c) Show that the coordinate time \( t \) it takes to fall from arbitrary \( r \) to just the “event horizon” at \( 2GM \) is, in contrast, infinite.

30. An incautious traveler has just passed the Schwarzschild radius, so he is now just inside \( r = 2GM \) and moving inwards, \( u^r < 0 \). Using the fact that \( u^a u_a = -1 \), show that even if he is allowed to accelerate, he can never stop falling in (i.e., he can’t have \( u^r = 0 \)), that the magnitude of \( |u^r| \) will have a minimum value as a function of \( r \), and find the maximum proper time before his world line terminates, i.e., he reaches \( r = 0 \).

31. Although four-velocity doesn’t exactly apply to photons, we can define an affine parameter \( \lambda \) along the photon’s path, and then define \( u^a = dx^a/d\lambda \). The geodesic equation for \( u^a \) is the same as usual \( du^a/d\lambda = -\Gamma^a_{\mu\nu} u^\mu u^\nu \), and therefore in the Schwarzschild metric, \( u_t = -E \) and \( u_\phi = L \) will still be conserved. Our goal in this problem is to find the cross-section for a photon to be absorbed by a black hole.
(a) Use the fact that \( u^a u_a = 0 \) for photons to find a formula for \( u^r \) for a photon as a function of \( r, E, L, \) and \( M \).
(b) The formula you just found should have \( (u^r)^2 = \infty \) at \( r = 0 \), then it should falls for a while, and then rises again to its ultimate value at \( r = \infty \). It therefore has a global minimum somewhere in between. Find the value of \( r \) where this occurs, and find the value of \( (u^r)^2 \) there, as a function of \( L, E, \) and \( M \).
(c) If the value you found in part (b) is positive, then \( u^r \) never vanishes, which means the photon continues all the way to the singularity at \( r = 0 \). If the value you found is negative, then it must have been zero somewhere, and therefore the photon must have turned around and left again. For what values of \( L \) is the photon absorbed?
(d) A photon comes in from infinity, such that it has an impact parameter of \( b \); that is, were it not for gravity, it would miss the black hole by a distance \( b \). What is the quantity \( L \) for this photon in terms of \( b \) and \( E \)? (this can be calculated far away, when gravity is negligible)? Find the cross section of the black hole for photons.