Problems 32-36

32. Rindler spacetime is a two-dimensional spacetime of considerable importance in the understanding of quantum field theory in curved spacetime. The metric is given by

\[ ds^2 = -x^2 dt^2 + dx^2 \]

(a) Find all non-zero components of the connection \( \Gamma^\alpha_{\mu\nu} \). Find the four-acceleration (actually, two-acceleration) \( A^\mu \) and its magnitude \( a A \) experienced by an observer staying at fixed position \( x \). Show that it diverges at the apparent coordinate singularity \( x = 0 \).

(b) Calculate a relationship between \( x \) and \( t \) for a light beam moving leftwards/rightwards. From this relationship, define two null coordinates \( v \) and \( w \). Write the metric in terms of these coordinates. The metric should look something like \( ds^2 = f(v,w) dv dw \).

(c) Define new coordinates \( v' = v'(v) \) and \( w' = w'(w) \) that makes the metric look as simple as possible. Show that in these new coordinates, the former coordinate singularity at \( x = 0 \) has disappeared.

33. Consider a particle moving radially in the FRW (Friedmann-Robertson-Walker) metric, so that \( u^\phi = u^\theta = 0 \). I recommend using the \( (t, \psi, \theta, \phi) \) coordinates

(a) Find an equation for how the radial velocity changes \( du^\nu / dt \) for a particle following a geodesic.

(b) Define \( p = mu^\nu \), where \( u^\nu \) is the radial velocity in orthonormal coordinates, \( m \) is the (constant) mass of the particle and \( p \) is the momentum. Show that \( ap \) is constant; that is, that \( da p / dt = 0 \).

34. Find the volume of the universe, defined as \( V = \int \sqrt{\gamma} d^3 r \), where \( \gamma \) is the determinant of the space part of the metric, for a closed universe \( (k = 1) \). I recommend using the \( (t, \psi, \theta, \phi) \) coordinates.

(a) In terms of the scale factor \( a \) at any time.

(b) At present, in terms of \( H_0 \) and \( \Omega \) (assuming \( \Omega > 1 \)).

35. Assume a flat universe \( k = 0 \) that is completely dominated by either radiation or matter. A photon leaves the origin at \( t = 0 \) and travels radially outwards. Find the distance the photon travels by time \( t \) for each of the two cases.

36. Suppose a universe contains only radiation, but does not necessarily have \( \Omega = 1 \). Find a relationship between the Hubble constant \( H_0 \) and the age of the universe in terms of \( \Omega \).