41. Write the formula for the size of gravitational waves from a circular Newtonian system in the form

\[ h \sim h_0 \left( \frac{M}{M_0} \right)^a \left( \frac{T}{T_0} \right)^b \left( \frac{r}{r_0} \right) \]

where \( M, T, \) and \( r \) are the total mass, period, and distance to the orbiting bodies, \( a \) and \( b \) are simple powers. Choose the distance \( r_0 \) such that \( h_0 \) is of order \( 10^{-21} \) to \( 10^{-22} \). Choose \( M_0 \) and \( T_0 \) as appropriate, for:

(a) Two white dwarfs of total mass \( 2 \times 1 M_\odot \), orbiting with a period appropriate for a separation around \( R = 4 \times 10^5 \) km.

(b) Two neutron stars of total mass \( 2 \times 1.4 M_\odot \), orbiting with a period appropriate for a separation around \( R = 100 \) km.

As derived in class, the general formula for the magnitude of the metric deviations in the \((r, \theta, \phi)\) basis is just

\[ h_{ij}^{TT} = \frac{2G\mu}{r} \left( GM\Omega \right)^{2/3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -h_0^0 & h_0^0 \\ 0 & h_0^0 & h_0^0 \end{pmatrix} \]

where \( h_r^0 = (1 + \cos^2 \theta) \cos \left[ 2(\phi - \Omega t + \Omega r) \right] \)

\( h_\theta^0 = 2 \cos \theta \sin \left[ 2(\phi - \Omega t + \Omega r) \right] \)

Let’s call the matrix part \( h_{ij}^0 \). We’ve arbitrarily chosen to take a factor of two out of the matrix, which means that \( h_{ij}^0 \) is never bigger than 2, and more typically around 1. The angular velocity can then be rewritten as \( \Omega = 2\pi / T \).

We need to work out an appropriate period for our two cases. This is most easily done by scaling. It is easy to see from class notes, or Kepler’s third law, that \( R^3 \propto MT^2 \). For a system with 1 solar mass (like the Earth around the Sun), we know that at 1 AU (150,000,000 km), the period is about one year. If we decrease the distance by a factor of 375 and increase the mass by about a factor of 2 (for the white dwarfs), this implies a period that is decreased by a factor of \( \sqrt{375^3 \cdot 2} \approx 10^4 \), which makes the period a bit less than an hour, so we’ll scale it to one hour. For the neutron star pair, the distance compared to Earth has been reduced by a factor of 1,500,000, and the mass increased by 2.8, so the period is decreased by a factor of \( \sqrt{(1.5 \times 10^6)^3 \cdot 2.8} \approx 3 \times 10^9 \), which reduced the period to about 10 ms. In each case, we’ll write \( \mu = (4\mu/M)^{1/2} M \), because the factor in parentheses will be less than or equal to one, and normally very close to one. We’ll always scale the distance to 1 pc = 3.086 \times 10^{16} \) m. For the white dwarfs, we have
\[ h^{TT}_y = \frac{2G\mu}{r} \left( \frac{2\pi GM}{T} \right)^{2/3} h^0_y = \left( \frac{4\mu}{M} \right) \left( \frac{GM}{2^{1/3} T^{2/3} r} \right) h^0_y \]

\[ = \frac{\pi^{2/3} (2 \times 1477 \text{ m})^{5/3} (4\mu/M) h^0_y}{2^{1/3} \left( (3600\text{s})(2.998 \times 10^8 \text{ m/s}) \right)^{2/3} (3.086 \times 10^{16} \text{ m})} \left( \frac{M}{2M_\odot} \right)^{5/3} \left( \frac{h}{T} \right)^{2/3} \left( \frac{\text{pc}}{r} \right) \]

\[ = 3.189 \times 10^{-19} h^0_y \left( \frac{4\mu}{M} \right) \left( \frac{M}{2M_\odot} \right)^{5/3} \left( \frac{h}{T} \right)^{2/3} \left( \frac{\text{pc}}{r} \right) \]

\[ = 3.189 \times 10^{-21} h^0_y \left( \frac{4\mu}{M} \right) \left( \frac{M}{2M_\odot} \right)^{5/3} \left( \frac{h}{T} \right)^{2/3} \left( \frac{100 \text{ pc}}{r} \right) \]

I do not know where the nearest white dwarf pair is, but there won’t be a whole lot of them within 100 pc.

\[ h^{TT}_y = \frac{\pi^{2/3} (2.8 \times 1477 \text{ m})^{5/3} (4\mu/M) h^0_y}{2^{1/3} \left( (0.01 \text{s})(2.998 \times 10^8 \text{ m/s}) \right)^{2/3} (3.086 \times 10^{16} \text{ m})} \left( \frac{M}{2.8M_\odot} \right)^{5/3} \left( \frac{10 \text{ ms}}{T} \right)^{2/3} \left( \frac{\text{pc}}{r} \right) \]

\[ = 2.827 \times 10^{-15} h^0_y \left( \frac{4\mu}{M} \right) \left( \frac{M}{2.8M_\odot} \right)^{5/3} \left( \frac{10 \text{ ms}}{T} \right)^{2/3} \left( \frac{\text{pc}}{r} \right) \]

\[ = 2.827 \times 10^{-21} h^0_y \left( \frac{4\mu}{M} \right) \left( \frac{M}{2.8M_\odot} \right)^{5/3} \left( \frac{10 \text{ ms}}{T} \right)^{2/3} \left( \frac{\text{Mpc}}{r} \right) \]

At a distance of 1 Mpc, we can see everything in our galaxy, and some members of the Local group, such as the Andromeda Galaxy (M31) are about 2 Mpc away.

42. Consider Lie derivatives.

(a) Show that the Lie derivative of a 1-form is given by \( \mathcal{L}_\xi \alpha_i = \xi^j \partial_j \alpha_i + \alpha_j \partial_i \xi^j \).

You may assume the Leibnitz rule applies; that is, \( \mathcal{L}_\xi (AB) = (\mathcal{L}_\xi A)B + A\mathcal{L}_\xi B \).

We start by considering the Lie derivative acting on the scalar product of a 1-form and a vector. We must have

\[ \xi^j \partial_j (\alpha_i \nu^i) = \mathcal{L}_\xi (\alpha_i \nu^i) = (\mathcal{L}_\xi \alpha_i) \nu^i + \alpha_i \mathcal{L}_\xi \nu^i = (\mathcal{L}_\xi \alpha_i) \nu^i + \alpha_i (\xi^j \partial_j \nu^i - \nu^j \partial_j \xi^i), \]

\[ \xi^j (\partial_j \alpha_i) \nu^i + \xi^i \alpha_i \partial_j \nu^i = (\mathcal{L}_\xi \alpha_i) \nu^i + \alpha_i \xi^j \partial_j \nu^i - \alpha_i \nu^j \partial_j \xi^i, \]

\[ (\mathcal{L}_\xi \alpha_i) \nu^i = (\xi^j \partial_j \alpha_i + \alpha_j \partial_i \xi^j) \nu^i. \]

Since this must be true for any vector field \( \nu^i \), we have

\[ \mathcal{L}_\xi \alpha_i = \xi^j \partial_j \alpha_i + \alpha_j \partial_i \xi^j \]
(b) Show that $\mathcal{L}_\xi v^i$ is a tensor. Hint: show that $\mathcal{L}_\xi v^i = \xi^i \nabla_j v^j - v^j \nabla_j \xi^i$. 

We start with the expression on the right, which is manifestly covariant, since it involves covariant derivatives. Expanding these covariant derivatives, we see that

$$
\xi^i \nabla_j v^j - v^j \nabla_j \xi^i = \xi^i \left( \partial_j v^j + v^k \Gamma^i_{kj} \right) - v^j \left( \partial_j \xi^i + \xi^k \Gamma^i_{kj} \right)
$$

$$
= \xi^i \partial_j v^j - v^j \partial_j \xi^i + \Gamma^i_{kj} \left( \xi^j v^k - \xi^k v^j \right) = \mathcal{L}_\xi v^i
$$

Hence we have proven it.

43. Show that the rate of loss of angular momentum from a Newtonian binary is given by

$$
d\mathcal{J} = \frac{32 \mu^2}{5M} \left( GM\Omega \right)^{2/3}
$$

Hint: You can write the angular momentum $J$ and energy $E$ in terms of the radius or angular velocity. Then you can use either of

$$
dJ = \left( \frac{dJ}{d\Omega} \right) \left( \frac{dE}{d\Omega} \right)^{-1} \left( \frac{dE}{dR} \right) = \left( \frac{dJ}{dR} \right) \left( \frac{dE}{dR} \right)^{-1} \left( \frac{dE}{dt} \right)
$$

The energy of the system is the potential energy plus the kinetic energy. The potential energy is $-\frac{Gm_1 m_2}{R} = -GM \mu/R$. The kinetic energy can be most easily worked out as $\frac{1}{2} \mu \left( R\Omega \right)^2$, or more carefully,

$$
T_1 + T_2 = \frac{m_1}{2} \left( \frac{m_2}{M} \Omega R \right)^2 + \frac{m_2}{2} \left( \frac{m_1}{M} \Omega R \right)^2 = \frac{m_1 m_2 \Omega^2 R^2}{2M} = \frac{1}{2} \mu \Omega^2 R^2.
$$

However, we know from Kepler’s Third law that $\Omega = \sqrt{GM/R^3}$, so this can be rewritten as $GM \mu/2R$, and the total energy is therefore $E = -GM \mu/2R$.

The angular momentum is $J = R\mu R\Omega$, or more carefully,

$$
J = \left( \frac{m_2}{M} R \right) m_1 \left( \frac{m_2}{M} \Omega R \right) + \left( \frac{m_1}{M} R \right) m_2 \left( \frac{m_1}{M} \Omega R \right) = \frac{m_1 m_2 \Omega R^2}{M} = \mu \Omega R^2.
$$

Using $\Omega = \sqrt{GM/R^3}$ again, this becomes $J = \mu \sqrt{GM R}$. We then use the second of our proposed identities to figure out how the two rates are related. We find
\[
\frac{dJ}{dt} = \left( \frac{dJ}{dR} \right) \left( \frac{dE}{dR} \right)^{-1} \left( \frac{dE}{dt} \right) = \left( \frac{\mu}{2} \sqrt{\frac{GM}{R}} \right) \left( \frac{GM \mu}{2R^2} \right)^{-1} \left( \frac{dE}{dt} \right) = \sqrt{\frac{R^3}{GM}} \left( \frac{dE}{dt} \right)
\]

\[
= \frac{1}{\Omega} \left[ \frac{32}{5} \frac{\mu^2}{GM^2} (GM\Omega)^{10/3} \right] = \frac{32}{5} \frac{\mu^2}{M} (GM\Omega)^{7/3}
\]