Solutions to Problems 44-47

44. In class, when demonstrating conservation of the stress-energy tensor, I used the relationship

\[ \frac{1}{2} T^{\mu\nu} \left( \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\alpha} \partial_\nu \xi^\alpha \right) = T^{\mu\nu} \nabla_\mu \xi_\nu \]

Fill in the missing steps.

\[ \frac{1}{2} T^{\mu\nu} \left( \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\alpha} \partial_\nu \xi^\alpha \right) = T^{\mu\nu} \left( \frac{1}{2} \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha \right) \]

\[ = T^{\mu\nu} \left( \frac{1}{2} \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \nabla_\mu \xi^\alpha - g_{\alpha\nu} \frac{1}{2} \xi^\alpha \right) \]

\[ = T^{\mu\nu} \left[ \frac{1}{2} \xi^\alpha \partial_\alpha g_{\mu\nu} + \nabla_\mu \xi^\alpha \right] - \frac{1}{2} \xi^\beta \left( \partial_\beta g_{\mu\nu} + \partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu} \right) \]

\[ = T^{\mu\nu} \nabla_\mu \xi_\nu \]

45. Show that the variation of the inverse metric is related to the variation of the metric by

\[ \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}. \]

The inverse metric is defined by \( g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \). For the perturbed metric, we have

\[ \delta \xi^\gamma = \left( g^{\alpha\beta} + \delta g^{\alpha\beta} \right) \left( g_{\beta\gamma} + \delta g_{\beta\gamma} \right) = \delta \xi^\gamma + \delta g_a^{\alpha\beta} \delta g_{\beta\gamma}, \]

\[ \delta g^{\alpha\beta} g_{\beta\gamma} = -g^{\alpha\beta} \delta g_{\beta\gamma}, \]

\[ \delta g_{\alpha\beta} g^\gamma_\gamma = -g^{\alpha\beta} g^\gamma_\gamma \delta g_{\beta\gamma}, \]

\[ \delta g_{\alpha\beta} = -g^{\alpha\beta} g^\gamma_\gamma \delta g_{\beta\gamma}. \]

46. The Lagrangian density for the electromagnetic field is

\[ \mathcal{L} = -\frac{1}{4} \varepsilon_0 F_{\mu\nu} F^{\mu\nu}, \]

where the fields \( F \) are defined as \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \) and \( F^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \). Show that the Euler-Lagrange equations yield the standard source-free Maxwell equations, \( \varepsilon_0 \nabla_\mu F^{\mu\nu} = 0 \).

\( \partial \mathcal{L} / \partial A_\mu \) vanishes, because \( A \) appears nowhere without a derivative. To find the other term, we take
Substituting this into the Euler-Lagrange equations, we have

\[ 0 = \frac{\partial L}{\partial A_v} = \nabla \frac{\partial L}{\partial \nabla A_v} = -\varepsilon_0 \nabla F^{\mu \nu} \]

Up to overall sign, this is identical to the formula we want.

47. Show that for the Lagrangian density in problem 46, the stress-energy tensor for the electromagnetic field is

\[ T^{\mu \nu} = \varepsilon_0 \left( F^\mu_a F^{\nu a} - \frac{1}{4} g^{\mu \nu} F^{ab} F_{ab} \right) \]

Our action is

\[ S[A,g] = -\frac{1}{4} \varepsilon_0 \int d^4x \sqrt{-g} g^{\mu \nu} g^{\beta \delta} F_{a \beta} F_{\gamma \delta} \]

Since we can write our fields \( F \) with ordinary derivatives, there are no metric elements lurking in \( F \) factors. We already know how to vary each of the factors that appears in this case, namely,

\[ \sqrt{-g} \rightarrow \sqrt{-g} \left( 1 + \frac{1}{4} g^{\mu \nu} \delta g_{\mu \nu} \right), \quad \text{and} \quad g^{\alpha \gamma} \rightarrow g^{\alpha \gamma} - g^{\mu \nu} g^{\beta \delta} \delta g_{\mu \nu} \]

Making these substitutions, we see that

\[ S[A,g + \delta g] = -\frac{1}{4} \varepsilon_0 \int d^4x \sqrt{-g} \left[ \left( 1 + \frac{1}{4} g^{\mu \nu} \delta g_{\mu \nu} \right) \left( g^{\alpha \gamma} - g^{\mu \nu} g^{\beta \delta} \delta g_{\mu \nu} \right) \right] \]

\[ = S[A,g] - \frac{1}{4} \varepsilon_0 \int d^4x \delta g_{\mu \nu} \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} F_{a \beta} F_{a \delta} - F_{a \beta} F_{a \delta} \right) \]

\[ = S[A,g] - \frac{1}{4} \varepsilon_0 \int d^4x \delta g_{\mu \nu} \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} F_{a \beta} F_{a \delta} - F_{a \beta} F_{a \delta} \right) \]

Comparing this with the previous equation, we see that

\[ T^{\mu \nu} = -\frac{1}{2} \varepsilon_0 \left( \frac{1}{2} g^{\mu \nu} F_{a \beta} F_{a \delta} - F_{a \beta} F_{a \delta} \right) = \varepsilon_0 \left( F^{\mu \delta} F_{\delta} - \frac{1}{4} g^{\mu \nu} F_{a \beta} F_{a \delta} \right) \]

This is sufficiently similar to the desired relation that we can stop.