1. The group SU(2) shows up in surprising places. Consider, for example, the two-dimensional harmonic oscillators, which can be written in the form

\[ H = \hbar \omega (a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \]

where \(a_1\) and \(a_2\) are two operators satisfying

\[ [a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \]

By conventional means, it is not hard to show that this Hamiltonian results in degenerate eigenvalues. But why? Is there a symmetry which results in this degeneracy?

(a) Define the three operators

\[ T_1 = \frac{1}{2} (a_1^\dagger a_2 + a_2^\dagger a_1), \quad T_2 = \frac{i}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \quad T_3 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2). \]

Show that these operators satisfy the SU(2) commutation relations,

\[ [T_a, T_b] = i \sum_c \epsilon_{abc} T_c. \]

This is three relations in all.

(b) Show that all three of the operators commute with the Hamiltonian

(c) There is a simple relationship between the Hamiltonian and the generators, namely

\[ T^2 = T_1^2 + T_2^2 + T_3^2 = \frac{1}{4} [H^2 / \hbar^2 \omega^2 - 1] \]

Demonstrating this is straightforward but laborious. Using this relationship, find the possible eigenvalues of \(H\), and their degeneracy, using only your knowledge of the eigenvalues of \(T^2\).