The original basis is:

\[
\begin{align*}
\mathbf{a} &= a\hat{x} \\
\mathbf{b} &= b(\cos \gamma \hat{x} + \sin \gamma \hat{y}) \\
\mathbf{c} &= c\hat{z}
\end{align*}
\]

An arbitrary element of the translation group is then \( n_1\mathbf{a} + n_2\mathbf{b} + n_3\mathbf{c} \), where \((n_1, n_2, n_3)\) are all integers, or instead we can choose \(n_1\) and \(n_3\) to be half-integers, but \(n_2\) is still an integer.

Imagine switching basis to

\[
\begin{align*}
\mathbf{T}_1 &= \frac{1}{2}(\mathbf{a} + \mathbf{c}) \\
\mathbf{T}_2 &= \mathbf{b} \\
\mathbf{T}_3 &= \frac{1}{2}(\mathbf{a} - \mathbf{c})
\end{align*}
\]

Then it is pretty easy to see that \( \mathbf{T}_1 \pm \mathbf{T}_3 \) yields the two vectors \( \mathbf{a} \) and \( \mathbf{c} \). It follows that

\[
n_1\mathbf{a} + n_2\mathbf{b} + n_3\mathbf{c} = n_1(\mathbf{T}_1 + \mathbf{T}_3) + n_2\mathbf{T}_2 + n_3(\mathbf{T}_1 - \mathbf{T}_2) = (n_1 + n_3)\mathbf{T}_1 + n_2\mathbf{T}_2 + (n_1 - n_3)\mathbf{T}_3
\]

Given the restrictions on \((n_1, n_2, n_3)\), it is obvious that all three of \((n_1 + n_3, n_2, n_1 - n_3)\) will be integers, and we have accomplished our goal.

The area of the parallelogram bounded by \( \mathbf{a} \) and \( \mathbf{b} \) is \( ab \sin \gamma \), which is the same as the magnitude of \( \mathbf{a} \times \mathbf{b} \). This must then be multiplied by the amount that \( \mathbf{c} \) sticks out of the plane of \( \mathbf{a} \) and \( \mathbf{b} \), so the total volume is \( V = |\mathbf{a} \times \mathbf{b}| \cos \theta \), where \( \theta \) is the angle between the vector \( \mathbf{c} \) and the perpendicular to \( \mathbf{a} \) and \( \mathbf{b} \). A little thought will convince you that this implies \( V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| \) , with the absolute value taking into account the fact that the cross product \( \mathbf{a} \times \mathbf{b} \) might point the opposite direction from \( \mathbf{c} \). We therefore have

\[
V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |ab \sin \gamma \hat{z} \cdot \mathbf{c}\hat{z}| = abc \sin \gamma
\]

\[
V' = |(\mathbf{T}_1 \times \mathbf{T}_2) \cdot \mathbf{T}_3| = \frac{1}{2} \left| (a\hat{x} + c\hat{z}) \times b(\cos \gamma \hat{x} + \sin \gamma \hat{y}) \right| \cdot (a\hat{x} - c\hat{z})|
\]

\[
= \frac{1}{2} b \left| (a \sin \gamma \hat{z} + c \cos \gamma \hat{y} - c \sin \gamma \hat{x}) \cdot (a\hat{x} - c\hat{z}) \right| = \frac{1}{2} abc \sin \gamma - \sin \gamma - \sin \gamma
\]

The minus sign inside the absolute value just indicates that the basis set \( \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \) is left-handed. If you want to, this can be fixed in numerous ways, such as changing the sign of \( \mathbf{T}_3 \). It is also easy to find the volume by taking the determinant of a matrix consisting of the three vectors.