Physics 745 - Group Theory
Solution Set 20

1. \([10]\) Prove, using only the commutation relations, the first three identities (2.9) from the notes, namely

\[
\left[T^2, T_a\right] = 0, \quad \left[T_i, T_a\right] = \pm T_a, \quad \text{and} \quad T^2 = T_+T_- + T_z^2 \pm T_3
\]

These are, in fact, three, two, and two identities respectively. On the first one, it is very helpful to use the identity \(\left[A^2, B\right] = A\left[A, B\right] + \left[A, B\right] A\).

It is simplest to write out the commutation relations we need to prove, and then work them out. The first one is the hardest.

\[
\left[T^2, T_1\right] = \left[T_1^2 + T_2^2 + T_3^2, T_1\right]
\]

\[
= T_1 [T_1, T_1] + [T_1, T_1] T_1 + T_2 [T_2, T_1] + [T_2, T_1] T_2 + T_3 [T_3, T_1] + [T_3, T_1] T_3
\]

\[
= 0 + 0 - iT_2 T_3 - iT_3 T_2 + iT_2 T_3 + iT_3 T_2 = 0,
\]

\[
\left[T^2, T_2\right] = \left[T_1^2 + T_2^2 + T_3^2, T_2\right]
\]

\[
= T_1 [T_1, T_2] + [T_1, T_2] T_1 + T_2 [T_2, T_2] + [T_2, T_2] T_2 + T_3 [T_3, T_2] + [T_3, T_2] T_3
\]

\[
= iT_1 T_3 + iT_3 T_1 + 0 + 0 - iT_2 T_3 - iT_3 T_2 = 0,
\]

\[
\left[T^2, T_3\right] = \left[T_1^2 + T_2^2 + T_3^2, T_3\right]
\]

\[
= T_1 [T_1, T_3] + [T_1, T_3] T_1 + T_2 [T_2, T_3] + [T_2, T_3] T_2 + T_3 [T_3, T_3] + [T_3, T_3] T_3
\]

\[
= -iT_1 T_3 + iT_2 T_3 + iT_3 T_2 + 0 + 0 = 0.
\]

The next two aren’t nearly as bad.

\[
[T_3, T_1] = [T_3, T_1 + iT_2] = iT_2 - i^2 T_3 = T_4,
\]

\[
[T_3, T_2] = [T_3, T_1 - iT_2] = iT_2 + i^2 T_3 = -T_4.
\]

The last one isn’t too difficult either.

\[
T_+ T_- + T_3^2 \pm T_3 = (T_1 + iT_2)(T_1 \pm iT_2) + T_3^2 \pm T_3 = T_1^2 \mp iT_1 T_2 \pm iT_3 T_2 + T_2^2 + T_3^2 \mp T_3
\]

\[
= T_1^2 \pm i[T_1, T_2] + T_2^2 + T_3^2 \pm T_3 = T^2 \pm i^2 T_3 \pm T_3 = T^2
\]

Done!