Physics 745 - Group Theory

Solution Set 21

1. [10] In class (or the notes), I gave explicit instructions for how to find the irreducible representations $T_a^{(j)}$. To demonstrate that you understand this, write explicitly $T_3^{(2)}$, $T_\pm^{(2)}$, $T_1^{(2)}$, and $T_2^{(2)}$ for the *j* = 2 irrep. Check that it is correct by computing

$$\mathbf{T}^2 = T_1^2 + T_2^2 + T_3^2$$

and show that it has the correct value.

The matrix T_3 is diagonal and takes on values from $+j$ to $-j$, so it is

$$T_3^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

The matrix $T_+^{(2)}$ has matrix elements $\langle m' | T_+^{(2)} | m \rangle = \delta_{m',m+1} \sqrt{j^2 + j - m^2 - m}$, where $j = 2$, so this is $\sqrt{6 - m^2 - m}$ in the diagonal just above the main diagonal. We then just Hermitian conjugate it to get $T_-^{(2)}$:

$$T_+^{(2)} = \begin{pmatrix} 0 & \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_-^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}$$

We can then get T_1 and T_2 from $T_1 = \frac{1}{2}(T_+ + T_-)$ and $T_2 = \frac{1}{2} i (T_- - T_+)$, which works out to

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$$

We still have to check this, which is done by working out $\mathbf{T}^2$.

$$\mathbf{T}^2 = T_1^2 + T_2^2 + T_3^2$$

$$= \begin{pmatrix} 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \frac{5}{2} & 0 & \frac{3}{2} & 0 \\ \sqrt{\frac{3}{2}} & 0 & 3 & 0 & \sqrt{\frac{3}{2}} \\ 0 & \frac{3}{2} & 0 & \frac{5}{2} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \frac{5}{2} & 0 & -\frac{3}{2} & 0 \\ -\sqrt{\frac{3}{2}} & 0 & 3 & 0 & -\sqrt{\frac{3}{2}} \\ 0 & -\frac{3}{2} & 0 & \frac{5}{2} & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

It should be a constant, and equal to $2^2 + 2 = 6$, and it does work out to this, so we are done.

2. [5] This problem has to do with breaking down an unknown representation of *SO*(3) into irreps.

(a) [2] Using the highest weight decomposition described in the notes, work out the decomposition of the defining generators of *SO*(3), given in equation (2.5):

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to the highest weight decomposition algorithm, we first need to find the eigenvalues of T_3, which we do by solving the characteristic equation $|T_3 - \lambda\mathbf{1}| = 0$, so

$$0 = \begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 - i^2\lambda = \lambda\left(1 - \lambda^2\right)$$

The three eigenvalues are then +1, 0, and -1. The highest weight is +1, which implies this must contain a copy of the representation (1). This representation should have eigenvalues +1, 0, and -1, so indeed, this is the representation. Note that we didn't need the other two matrices, we need only look at T_3 to determine everything.

(b) [3] A certain representation of $SO(3)$ has generators given by

$$T_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad T_2 = \frac{i}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

How does this break down into irreps?

We proceed exactly as before. It helps that the matrix is block diagonal, but even without that, it isn't too bad

$$0 = \begin{vmatrix} \tfrac{1}{2}-\lambda & \tfrac{1}{2} & 0 & 0 \\ \tfrac{1}{2} & \tfrac{1}{2}-\lambda & 0 & 0 \\ 0 & 0 & -\tfrac{1}{2}-\lambda & \tfrac{1}{2} \\ 0 & 0 & \tfrac{1}{2} & -\tfrac{1}{2}-\lambda \end{vmatrix}$$

$$= \left(\tfrac{1}{2}-\lambda\right)\begin{vmatrix} \tfrac{1}{2}-\lambda & 0 & 0 \\ 0 & -\tfrac{1}{2}-\lambda & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} & -\tfrac{1}{2}-\lambda \end{vmatrix} - \tfrac{1}{2}\begin{vmatrix} \tfrac{1}{2} & 0 & 0 \\ 0 & -\tfrac{1}{2}-\lambda & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} & -\tfrac{1}{2}-\lambda \end{vmatrix}$$

$$= \left[\left(\tfrac{1}{2}-\lambda\right)^2 - \left(\tfrac{1}{2}\right)^2\right]\begin{vmatrix} -\tfrac{1}{2}-\lambda & \tfrac{1}{2} \\ \tfrac{1}{2} & -\tfrac{1}{2}-\lambda \end{vmatrix} = \left[\left(\tfrac{1}{2}-\lambda\right)^2 - \left(\tfrac{1}{2}\right)^2\right]\left[\left(-\tfrac{1}{2}-\lambda\right)^2 - \left(\tfrac{1}{2}\right)^2\right]$$

$$= \left(\lambda^2 - \lambda\right)\left(\lambda^2 + \lambda\right) = \lambda^2\left(\lambda - 1\right)\left(\lambda + 1\right)$$

This has roots of +1, -1, 0 and 0. Since the highest eigenvalue is +1, it must once again include the (1) representation, which accounts for the eigenvalues +1, 0, and -1. This leaves the 0 eigenvalue unaccounted for, but this is just the eigenvalue for the (0) representation, so our final answer is $(1) \oplus (0)$.